MTH 266, Proof Set #3, Fall 2018 Name ____

Instructions: For each statement listed below write up a complete proof, following the guidelines of the proof writing handout given out in class, and the examples in the appendix in our textbook. You must include **words** explaining your reasoning, citing relevant theorems or definitions, not just computation and symbols. Proofs will be graded both on accuracy and style. Proofs may be handwritten **legibly**, or else they should be typed using a word processor equipped with an equation editor (you will asked to do a second draft of these proofs, and you may find it easier to edit if you type it up the first time). Attach the proofs to this page. You may use examples in the textbook, or those you find online, for pointers and direction, but the proofs you write should be **your own words** and **notation consistent with our textbook**.

Required: Do all the proofs in this section.

1. Use mathematical induction to prove that if the standard matrices for the linear transformations $T_1, T_2, T_3, \dots T_n$ are the matrices $A_1, A_2, A_3, \dots A_n$ respectively, then the standard matrix for the composition $T(\vec{v}) = T_n\left(T_{n-1}\left(T_{n-2}\left(\dots\left(T_2\left(T_1(\vec{v})\right)\right)\right)\right)\right)$ is represented by $A = A_nA_{n-1}\dots A_2A_1$.

2. Prove the properties of the dot product for generic vectors $\vec{u}, \vec{v}, \vec{w}\ in\ R^3$ and c a scalar.
   a. $\vec{u}\cdot\vec{v} = \vec{v}\cdot\vec{u}$
   b. $(c\vec{u})\cdot\vec{v} = c(\vec{u}\cdot\vec{v}) = \vec{u}\cdot(c\vec{v})$
   c. $(\vec{u}+\vec{v})\cdot\vec{w} = \vec{u}\cdot\vec{w}+\vec{v}\cdot\vec{w}$
   d. $\vec{u}\cdot\vec{u} \geq 0$ and $\vec{u}\cdot\vec{u} = 0$ iff $\vec{u} = \vec{0}$

3. Suppose that $\vec{y}$ is orthogonal to both $\vec{u}\ and\ \vec{v}$. Prove that $\vec{y}$ is orthogonal to $\vec{u}+\vec{v}$.

4. Show that if $\vec{x}$ is in both W and $W^{\perp}$, then $\vec{x} = \vec{0}$.

5. Let W be a subspace of R^n with orthogonal basis $\{\overrightarrow{w_1}, \dots, \overrightarrow{w_p}\}$ and let $\{\overrightarrow{v_1}, \dots, \overrightarrow{v_q}\}$ be an orthogonal basis for $W^{\perp}$. Prove that $\{\overrightarrow{w_1}, \dots, \overrightarrow{w_p}, \overrightarrow{v_1}, \dots, \overrightarrow{v_q}\}$ is an orthogonal basis for R^n.

6. Prove that if a linear transformation $T: V \to W$ is onto, then the dimension of W cannot be greater than the dimension of V.

7. Let T be a linear transformation that maps R^n onto R^n. Show that T^{-1} exists and maps R^n onto R^n. Is T^{-1} also one-to-one? If so, prove it. If not, prove that instead.

8. Prove that $\|\vec{u}+\vec{v}\| = \|\vec{u}\|+\|\vec{v}\|$ if and only if $\vec{v} = k\vec{u}$.

9. Prove that $\|\vec{u}+\vec{v}\|^2 + \|\vec{u}-\vec{v}\|^2 = 2\|\vec{u}\|^2 + 2\|\vec{v}\|^2$.

Options: Do at least two proofs from this section.

10. Let $T: V \to W$ be linear transformation from a vector space V into a vector space W. Prove that the range of T is a subspace of W. Prove that the range of T is a subspace of W. [Hint: typical elements of the range of T have the form $T(\vec{x}), T(\vec{y})\ for\ \vec{x}, \vec{y}$ in V.

11. Prove that $P^{-1} = P^T$ if and only if the columns of P for an orthonormal basis for R^n for $P\ n \times n$.

12. An *affine transformation* $T: R^n \to R^m$ has the form $T(\vec{x}) = A\vec{x} + \vec{b}$ with A an *m*x*n* matrix and $\vec{b}$ in R^m. Show that T is *not* a linear transformation when $\vec{b} \neq \vec{0}$.