MTH 266, Proof Set #2, Fall 2018 Name ____

Instructions: For each statement listed below write up a complete proof, following the guidelines of the proof writing handout given out in class, and the examples in the appendix in our textbook. You must include **words** explaining your reasoning, citing relevant theorems or definitions, not just computation and symbols. Proofs will be graded both on accuracy and style. Proofs may be handwritten **legibly**, or else they should be typed using a word processor equipped with an equation editor (you will asked to do a second draft of these proofs, and you may find it easier to edit if you type it up the first time). Attach the proofs to this page. You may use examples in the textbook, or those you find online, for pointers and direction, but the proofs you write should be **your own words** and **notation consistent with our textbook**.

Required: Do all the proofs in this section.

1. Given a generic 2x2 matrix A, and a second 2x2 matrix C, show the following:
   a. If you exchange two rows of A to produce matrix B, then $\det(A) = -\det(B)$.
   b. $\det A^T = \det A$
   c. $|AC| = |A||C|$
   d. Using property c above, show that $\det A^k = (\det A)^k$ for some natural number k.
   e. Show that $\det rA = r^2 \det A$, for some scalar r.

2. Prove that if AB is invertible then both A, B must be invertible.

3. Prove for generic vectors $\vec{u}, \vec{v}, \vec{w}\ in\ R^3$ and c, d in R the algebraic properties of vectors. [Hint: each vector property can be reduced to properties of real numbers and definitions. These should all be relatively short. Some will be *very* short.]
   a. $\vec{u} + \vec{v} = \vec{v} + \vec{u}$
   b. $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$
   c. $c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v}$
   d. $(c + d)\vec{u} = c\vec{u} + d\vec{u}$

4. Show that the set of all polynomials of degree at most 3 forms a subspace of P_n.

5. Determine if $S = \left\{ \begin{bmatrix} a & a^2 & a^3 \\ a^2 & a & a^2 \\ a^3 & a^2 & a \end{bmatrix} \middle| a\ is\ real \right\}$ is a subspace of $M_{3\times 3}$. If it is, prove it; if it is not, find a counterexample.

6. Prove that in the set of real-valued functions $\{\sin t, \sin 2t, \sin t \cos t\}$ is *not* a basis for the space spanned by the set. Then find the basis for that space. [Hint: can one of the functions in the set be written as a linear combination of the others?]

7. Prove that $p_1(t) = 1 + t^2, p_2(t) = t - 3t^2, p_3(t) = 1 + t - 3t_2$ is a basis for P_2.

Options: Do at least one proof from this section.

8. Prove that there is an equivalent set of properties of determinants for column operations as there are for row operations. Complete the proof for each property on 3x3 matrices. [Hint: you

may find using elementary matrices to be of help. To perform column operations with elementary matrices, multiply on the right instead of the left.]

a. Exchanging two rows of a matrix changes the sign of the determinant.
b. Adding two rows of a matrix does not change the value of the determinant.
c. Multiplying one row of a matrix by the scalar k changes the value of the determinant by k.

9. Take any three conditions from the Invertible Matrix Theorem and show that they are equivalent. [Note: equivalent means "if and only if" statements. The shortest strategy is to prove that $A \to B \to C \to A$.] You may appeal to smaller theorems of individual relationships in the textbook to help connect the steps. Cite the theorem number (or page) in your proof.

10. For the alternate definitions of addition or scalar multiplication on R^3, determine whether the resulting set is a vector space (prove your conclusion). You should check all properties. Be sure to state clearly any properties that fail.

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 + 1 \\ y_1 + y_2 + 1 \\ z_1 + z_2 + 1 \end{bmatrix}, c\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} cx_1 + c - 1 \\ cy_1 + c - 1 \\ cz_1 + c - 1 \end{bmatrix} \quad in\ R^3$$

The ten properties of a vector space are:

i. $\vec{u} + \vec{v}$ in V for every $\vec{u}, \vec{v} \in V$
ii. $\vec{u} + \vec{v} = \vec{v} + \vec{u}$
iii. $\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$
iv. V has a vector $\vec{0}$ such that for every $\vec{u}$ in V, $\vec{u} + \vec{0} = \vec{u}$
v. For every $\vec{u}$ in V, there is a vector in V denoted by $-\vec{u}$ such that $\vec{u} + (-\vec{u}) = \vec{0}$
vi. $c\vec{u}$ in V for every $\vec{u}$ in V, and every $c \in R$
vii. $c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v}$
viii. $(c + d)\vec{u} = c\vec{u} + d\vec{u}$
ix. $(cd)\vec{u} = c(d\vec{u})$
x. $1\vec{u} = \vec{u}$