MTH 266, Homework #6, Fall 2018 Name ____

Instructions: Write your work up neatly and attach to this page. Use exact values unless specifically asked to round. Show all work.

1. For each pair of vectors in i-iii, find the following:
   a. $\vec{u} \cdot \vec{v}$
   b. $\frac{\vec{u}}{\|\vec{u}\|}$ and $\frac{\vec{v}}{\|\vec{v}\|}$
   c. $\|\vec{u} + \vec{v}\|^2$
   d. $\|\vec{u} - \vec{v}\|$
   e. $\|\vec{u}\|$ and $\|\vec{v}\|$
   f. $\|\vec{u}\|^2 + \|\vec{v}\|^2$
   g. $\frac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}} \vec{v}$

   i. $\vec{u} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \vec{v} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$

   ii. $\vec{u} = \begin{bmatrix} 3 \\ 2 \\ -5 \\ 0 \end{bmatrix}, \vec{v} = \begin{bmatrix} -4 \\ 1 \\ -2 \\ 6 \end{bmatrix}$

   iii. $\vec{u} = \begin{bmatrix} 12 \\ 3 \\ -5 \end{bmatrix}, \vec{v} = \begin{bmatrix} 2 \\ -3 \\ 3 \end{bmatrix}$

2. Use the information in problem #1, for each pair of vectors in i-iii, to determine the following:
   a. The distance between $\vec{u}$ *and* $\vec{v}$.
   b. Are the vectors $\vec{u}$ *and* $\vec{v}$ orthogonal?
   c. What is the angle between $\vec{u}$ *and* $\vec{v}$?
   d. Find unit vectors in the direction of $\vec{u}$ *and* $\vec{v}$.
   e. The orthogonal projection of $\vec{u}$ in the direction of $\vec{v}$.
   f. If $\vec{u}$ *and* $\vec{v}$ are orthogonal, call the subspace spanned by the vectors W and find an orthonormal basis for the subspace.
   g. Find $W^{\perp}$ for $W = span\{\vec{u}, \vec{v}\}$.

3. Separate $\vec{y} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$ into $\vec{y}_{\parallel}$ *and* $\vec{y}_{\perp}$ if $\vec{y}_{\parallel}$ is in the direction of $\vec{u} = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$.

4. For each statement, indicate whether it's true or false. For the ones that are false, state the correct true statement.
   a. Not every linearly independent set in $\mathbb{R}^n$ is an orthogonal set.

   b. If $\vec{y}$ is a linear combination of nonzero vectors from an orthogonal set, then the weights in the linear combination can be computed without row operations on a matrix.

   c. If the vectors in an orthogonal set are normalized, then some of the new vectors may not be orthogonal.

   d. If the columns of an mxn matrix A are orthonormal, then the linear mapping $\vec{x} \longmapsto A\vec{x}$ preserves lengths.

   e. An orthogonal matrix is invertible.

f. If $\vec{x}$ is orthogonal to $\vec{u}_1$ and $\vec{u}_2$ and if W = Span{**u**$_1$, **u**$_2$}, then $\vec{x}$ must be in W$^{\perp}$.

g. If $\vec{y}$ is in a subspace W, then the orthogonal projection of $\vec{y}$ onto W is $\vec{y}$ itself.

h. If the columns of an nxp matrix U are orthonormal, then UU$^T\vec{y}$ is the orthogonal projection of $\vec{y}$ onto the column space of U.

i. In the Orthogonal Decomposition Theorem, each term in the formula $\vec{y}_{\parallel} = \sum_{i=1}^{p} \frac{\vec{y}\cdot\vec{u}_i}{\vec{u}_i\cdot\vec{u}_i}\overrightarrow{u_i}$ is itself an orthogonal projection of $\vec{y}$ onto a subspace of W.

j. The range of a linear transformation is a vector space.

k. If A is a 3x5 matrix and T is a linear transformation defined by $T(\vec{x}) = A\vec{x}$, then domain of T is R^3.

l. Every linear transformation is a matrix transformation and every matrix transformation is a linear transformation.

m. A linear transformation preserves the operations of vector addition and scalar multiplication.

n. A linear transformation $T: R^n \to R^m$ is completely determined by its effects on the columns of the *n*x*n* identity matrix.

o. When two linear transformations are performed one after another, the combined effect may not always be a linear transformation.

p. If A is a 3x2 matrix, then the transformation $\vec{x} \mapsto A\vec{x}$ cannot be one-to-one.

q. The columns of the standard matrix for a linear transformation from $T: R^n \to R^m$ are the images of the columns of the *n*x*n* identity matrix under T.

r. The best approximation to $\vec{y}$ by elements of a subspace W is given by the vector $\vec{y} - proj_W\vec{y}$.

s. The general least-squares problem is to find an $\vec{x}$ that makes $A\vec{x}$ as close to $\vec{b}$ as possible.

t. Any solution of $A^T A\vec{x} = A^T\vec{b}$ is a least-squares solution of $A\vec{x} = \vec{b}$.

u. If the columns of A are linearly independent, then the equation $A\vec{x} = \vec{b}$ has exactly one least-squares solution.

v. A least-squares solution of $A\vec{x} = \vec{b}$ is the point in the column space of A closest to $\vec{b}$.

5. How many rows and columns must a matrix have in order to define a mapping from R^5 into R^7 by the rule $T(\vec{x}) = A\vec{x}$.

6. Use the standard inner product on R^n to determine if the vectors or functions are orthogonal.

a. $\left\{\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\-1\end{bmatrix}\right\}$

c. $\left\{\begin{bmatrix}-1\\2\\1\\0\\0\end{bmatrix}, \begin{bmatrix}-1\\-3\\0\\5\\1\end{bmatrix}, \begin{bmatrix}0\\1\\2\\-1\\0\end{bmatrix}\right\}$

b. Find two more vectors orthogonal to $\begin{bmatrix}1\\-3\\5\\4\end{bmatrix}$

7. Show that the vectors $\vec{u}_1 = \begin{bmatrix}3\\-3\\0\end{bmatrix}, \vec{u}_2 = \begin{bmatrix}2\\2\\-1\end{bmatrix}, \vec{u}_3 = \begin{bmatrix}1\\1\\4\end{bmatrix}$ form an orthogonal basis for $\mathbb{R}^3$. Make this basis an orthonormal basis, and then use that basis to find the representation of $\vec{x} = \begin{bmatrix}5\\-3\\1\end{bmatrix}$ in that basis using the formula $\vec{x} = c_1\vec{u}_1 + c_2\vec{u}_2 + c_3\vec{u}_3 \; where \; c_j = \frac{\vec{x}\cdot\vec{u}_j}{\vec{u}_j\cdot\vec{u}_j}$ $(j = 1,2,3)$.

8. Verify that the given set of vectors $\{u_1,\ldots,u_n\}$ is orthogonal, and then write $\vec{x}$ as a pair of vectors $\overrightarrow{x_\parallel} \; and \; \overrightarrow{x_\perp}$, with W defined as the span of the specified vectors and $\overrightarrow{x_\parallel}$ in W. What is the best approximation to $\vec{x}$ in W? What is the distance from the subspace to the point $\vec{x}$.

a. $\vec{u}_1 = \begin{bmatrix}1\\2\\1\\1\end{bmatrix}, \vec{u}_2 = \begin{bmatrix}-2\\1\\-1\\1\end{bmatrix}, \vec{u}_3 = \begin{bmatrix}1\\1\\-2\\-1\end{bmatrix}, \vec{u}_4 = \begin{bmatrix}-1\\1\\1\\-2\end{bmatrix}, W = Span\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}, \vec{x} = \begin{bmatrix}4\\5\\-3\\3\end{bmatrix}$

b. $\vec{u}_1 = \begin{bmatrix}1\\1\\0\end{bmatrix}, \vec{u}_2 = \begin{bmatrix}-1\\1\\0\end{bmatrix}, W = Span\{\vec{u}_1, \vec{u}_2\}, \vec{x} = \begin{bmatrix}-1\\4\\3\end{bmatrix}$

c. $\vec{u}_1 = \begin{bmatrix}1\\1\\0\\-1\end{bmatrix}, \vec{u}_2 = \begin{bmatrix}1\\0\\1\\1\end{bmatrix}, \vec{u}_3 = \begin{bmatrix}0\\-1\\1\\-1\end{bmatrix}, W = Span\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}, \vec{x} = \begin{bmatrix}3\\4\\5\\6\end{bmatrix}$

9. Let T be defined by $T(\vec{x}) = A\vec{x}$. Find a vector $\vec{x}$ whose image under T is $\vec{b}$, and determine whether $\vec{x}$ is unique.

a. $A = \begin{bmatrix}1 & -2 & 3\\2 & -5 & 6\end{bmatrix}, \vec{b} = \begin{bmatrix}-6\\-4\end{bmatrix}$

b. $A = \begin{bmatrix}1 & -3 & 2\\3 & -8 & 8\\0 & 1 & 2\\1 & 0 & 8\end{bmatrix}, \vec{b} = \begin{bmatrix}1\\6\\3\\10\end{bmatrix}$

10. Find the least-squares approximation for $A\vec{x} = \vec{b}$.

a. $A = \begin{bmatrix}1 & 1 & 0\\1 & 1 & 0\\1 & 0 & 1\\1 & 0 & 1\end{bmatrix}, \vec{b} = \begin{bmatrix}1\\3\\8\\2\end{bmatrix}$

b. $A = \begin{bmatrix}2 & 1\\-2 & 0\\2 & 3\end{bmatrix}, \vec{b} = \begin{bmatrix}-5\\8\\1\end{bmatrix}$

11. Find the best-fit equation specified for the given set of data.
    a. $\{(1,0),(2,1),(4,2),(5,3)\}, y = \beta_0 + \beta_1 x$
    b. $\{(1,0),(2,1),(4,2),(5,3)\}, y = \beta_0 + \beta_1 x + \beta_2 x^2$
    c. $\{(4,1.58),(6,2.08),(8,2.5),(10,2.8),(12,3.1),(14,3.4),(16,3.8),(18,4.32)\}, y = \beta_1 x + \beta_2 x^2 + \beta_3 x^3$
12. Explain why regression on the equation $y = A\cos x + B\sin x$ is a problem we can solve using linear algebra, but $y = A\sin(Bx + C) + D$ is not.
13. Give four examples of equations which are not linear (in the sense that the functions are not linear in x), but which can be solved by linear algebra (linear in its coefficients) or which is intrinsically linear (it can be made linear through substitution or transformation of the variables).