MTH 266, Exam #2, Part I, Fall 2018 Name KEY

Instructions: Show all work. Give exact answers unless specifically asked to round. If you do not show work, problems will be graded as "all or nothing" for the answer only; partial credit will not be possible and any credit awarded for the work will not be available. On this portion of the exam, you may **NOT** use a calculator.

1. Compute the determinant by the cofactor method. (10 points)

$$\begin{vmatrix} 0 & 3 & 9 & 5 \\ 1 & 0 & -2 & 4 \\ -3 & 2 & 1 & 6 \\ 0 & 8 & 2 & 3 \end{vmatrix} = -3\begin{vmatrix} 1 & -2 & 4 \\ -3 & 1 & 6 \\ 0 & 2 & 3 \end{vmatrix} + 9\begin{vmatrix} 1 & 0 & 4 \\ -3 & 2 & 6 \\ 0 & 8 & 3 \end{vmatrix} - 5\begin{vmatrix} 1 & 0 & -2 \\ -3 & 2 & 1 \\ 0 & 8 & 2 \end{vmatrix}$$

$$= -3\left[1\begin{vmatrix} 1 & 6 \\ 2 & 3 \end{vmatrix} - (-3)\begin{vmatrix} -2 & 4 \\ 2 & 3 \end{vmatrix}\right] + 9\left[1\begin{vmatrix} 2 & 6 \\ 8 & 3 \end{vmatrix} - (-3)\begin{vmatrix} 0 & 4 \\ 8 & 3 \end{vmatrix}\right] - 5\left[1\begin{vmatrix} 2 & 1 \\ 8 & 2 \end{vmatrix} + 3\begin{vmatrix} 0 & -2 \\ 8 & 2 \end{vmatrix}\right]$$

$$= -3[(3-12) + 3(-6-8)] + 9[(6-48) + 3(0-32)] - 5[(4-8) + 3(0+16)]$$

$$= -3[-9 + 3(-14)] + 9[-42 + 3(-32)] - 5[-4 + 3(16)] =$$

$$= -3[-9-42] + 9[-138] - 5[-4+48] =$$

$$-3[-51] + 9[-138] - 5[44] = 153 - 1242 - 220$$

$$= -1309$$

2. Compute the determinant by using row operations. (8 points)

$$\begin{vmatrix} 1 & -1 & 5 \\ 2 & 4 & -3 \\ 3 & -2 & 0 \end{vmatrix}$$

$-2R_1 + R_2 \to R_2$ (no change)

$-3R_1 + R_3 \to R_3$ (no change)

$$\begin{vmatrix} 1 & -1 & 5 \\ 0 & 6 & -13 \\ 0 & 1 & -15 \end{vmatrix} = \begin{vmatrix} 6 & -13 \\ 1 & -15 \end{vmatrix} = 6(-15) - 1(-13) = -90 + 13$$

$$= -77$$

3. Determine if the following sets are linearly independent or dependent. Justify your answers **without performing matrix calculations.** (3 points each)

a. $\left\{ \begin{bmatrix} 1 \\ 7 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 6 \\ 5 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ 11 \\ 17 \end{bmatrix} \right\}$

not independent
(dependent)
more vectors than dimensions
4 vectors in $\mathbb{R}^3$ always dependent

b. $\left\{ \begin{bmatrix} 1 \\ 7 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} \right\}$

independent
2 vectors that are not multiples of each other must be independent

4. Given that A and B are $n \times n$ matrices with $\det A = 3$ and $\det B = -4$, find the following. (3 points each)

a) $det\ AB$ -12

d) $det\ A^T$ 3

b) $det\ A^{-1}$ $\frac{1}{3}$

e) $det\ 2A$ $2^n \cdot 3$

c) $det\ (-AB^2)$

$(-1)^n (3)(-4)^2$

$= (-1)^n\, 48$

MTH 266, Exam #2, Part II, Fall 2018 Name KEY

Instructions: Show all work. Give exact answers unless specifically asked to round. All complex numbers should be stated in standard form, and all complex fractions should be simplified. If you do not show work, problems will be graded as "all or nothing" for the answer only; partial credit will not be possible and any credit awarded for the work will not be available. On this portion of the exam, you *may* use a calculator to perform elementary matrix operations. Support your answers with work (reproduce the reduced matrices from your calculator) or other justification for full credit.

1. Determine if each statement is True or False. (1 point each)

a. T F If matrix B is formed by multiplying a row of matrix A by 4, then $det\ B = 4\ det\ A$

b. T F The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution if and only if there are no free variables.

c. T F If an $m \times n$ matrix has a pivot in every row, then the equation $A\mathbf{x} = \mathbf{b}$ has a unique solution for each **b** in R^m. not necessarily unique

d. T F If $\{\mathbf{u},\ \mathbf{v},\ \mathbf{w}\}$ is linearly independent, then **u**, **v**, and **w** are not in R^2.

e. T F If *A* and *B* are $m \times n$ matrices, then both AB^{T} and $A^{\mathrm{T}}B$ are defined.
$(m \times n)(n \times m)$ $(n \times m)(m \times n)$

f. T F If two rows of a 3×3 matrix *A* are the same, then det *A* = 0.

g. T F If $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is linearly independent, then so is $\{\mathbf{v}_1, \ldots, \mathbf{v}_{p-1}\}$.

h. T F The pivot columns of a matrix are always linearly dependent.

i. T F The rank of a matrix is defined by the dimension of the ~~null~~ col. space.

j. T F If det A is zero, then two rows or two columns of A are the same, or a row or a column is zero.
one row could also be a linear comb. of 2 or more other rows

k. T F If A and B are row equivalent, then their column spaces are the same.
Same dimension, but not identical

l. T F The vector space $\mathbb{P}_3$ and R^3 are isomorphic.
P_3 isomorphic to $\mathbb{R}^4$

m. T F A linearly independent set in a subspace H is a basis for H. also span space

n. T F If P_B is the change-of-coordinates matrix, then $\left[\vec{x}\right]_B = P_B\vec{x}$ for $\vec{x}$ in V.
$P_B[\vec{x}]_B = \vec{x}$

~~o.~~ T F The equilibrium vector for a stochastic matrix is always unique.

2. Determine if the columns of $A = \begin{bmatrix} 2 & 5 & 2 \\ 0 & 4 & -1 \\ 3 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$ form a linearly independent set and justify your answer. (6 points)

rref $\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ Since the reduced form has a pivot in every column so the vectors are independent

3. Given $T : \mathbb{R}^3 \to \mathbb{R}^3$ such that $T(\mathbf{x}) = \begin{bmatrix} 3x_1 - 2x_2 + x_3 \\ x_2 + 4x_3 \\ -2x_1 + 3x_3 \end{bmatrix}$ answer the following.

a. Find the standard matrix, *A*, such that $T(\mathbf{x}) = A\mathbf{x}$. (4 points)

$$A = \begin{bmatrix} 3 & -2 & 1 \\ 0 & 1 & 4 \\ -2 & 0 & 3 \end{bmatrix}$$

b. Is *T* onto $\mathbb{R}^3$? Justify your answer. (3 points)

rref $\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ pivot in every row, yes

c. Is T one-to-one? Justify your answer. (3 points)

yes, pivot in every column

4. Determine if the set $H = \left\{ \begin{bmatrix} 1 \\ 7 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 6 \\ 5 \\ 1 \end{bmatrix} \right\}$ forms a basis for $\mathbb{R}^3$. Justify your answer.

(6 points)

$rref \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

yes. vectors are both independent and span $\mathbb{R}^3$, so it forms a basis for $\mathbb{R}^3$

5. Assume that $A = \begin{array}{c} \begin{array}{ccccc} \downarrow & & \downarrow & & \downarrow \end{array} \\ \begin{bmatrix} 1 & 2 & -5 & 11 & -3 \\ 2 & 4 & -5 & 15 & 2 \\ 1 & 2 & 0 & 4 & 5 \\ 3 & 6 & -5 & 19 & -2 \end{bmatrix} \end{array}$ and $B = \begin{bmatrix} \underline{1} & 2 & 0 & 4 & 5 \\ 0 & 0 & \underline{5} & -7 & 8 \\ 0 & 0 & 0 & 0 & \underline{-9} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ are row equivalent.

a. Find a basis for the column space of A and state the dimension of $Col\ A$. (4 points)

$$Col\ A = \text{Span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -5 \\ -5 \\ 0 \\ -5 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \\ 5 \\ -2 \end{bmatrix} \right\}$$

$\dim Col\ A = 3$

b. Find a basis for the null space of A and state the dimension of $Nul\ A$. (6 points)

working from B

$x_1 + 2x_2 + 4x_4 + 5\cancel{x_5} = 0$

$5x_3 - 7x_4 + 8\cancel{x_5} = 0$

$-9x_5 = 0$

x_2, x_4 free

$x_5 = 0$

$\Rightarrow$

$x_1 = -2x_2 \quad -4x_4$

$x_2 = x_2$

$x_3 = \frac{7}{5}x_4$

$x_4 = x_4$

$x_5 = 0$

$$\vec{x} = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} -4 \\ 0 \\ 7/5 \\ 1 \\ 0 \end{bmatrix} x_4$$

$$Nul\ A = \text{Span}\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 7/5 \\ 1 \\ 0 \end{bmatrix} \right\}$$

c. Determine if $\mathbf{b} = \begin{bmatrix} -2 \\ 0 \\ 4 \\ -3 \end{bmatrix}$ is in $Col\ A$. Show appropriate work to justify your answer.

(4 points)

it is not in Col A

Since $\left[\begin{array}{ccc|c} 1 & -5 & -3 & -2 \\ 2 & -5 & 2 & 0 \\ 1 & 0 & 5 & 4 \\ 3 & 5 & -2 & -3 \end{array}\right] \Rightarrow \text{rref} \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$

which is inconsistent

d. What is the rank of A? (2 points)

3

6. Suppose matrix A is a 6×8 matrix with 5 pivot columns. Determine the following. (12 points)

$dim\ Col\ A$ = 5

$dim\ Nul\ A$ = 8-5 = 3

$dim\ Row\ A$ = 5

If $Col\ A$ is a subspace of R^m, then m= 6

$Rank\ A$ = 5

If $Nul\ A$ is a subspace of R^n, then n = 8

7. Given the bases $B = \{b_1, b_2, b_3\}$ and $C = \{c_1, c_2, c_3\}$ below, find the change of basis matrices $\underset{C\leftarrow B}{P}$ and $\underset{B\leftarrow C}{P}$. If the B-coordinate vector for $\vec{x}$ is as shown, find the C-coordinate vector for $\vec{x}$. (10 points)

$$\vec{b_1} = \begin{bmatrix}1\\1\\3\end{bmatrix}, \vec{b_2} = \begin{bmatrix}2\\0\\8\end{bmatrix}, \vec{b_3} = \begin{bmatrix}1\\-1\\3\end{bmatrix}, \vec{c_1} = \begin{bmatrix}2\\-1\\4\end{bmatrix}, \vec{c_2} = \begin{bmatrix}1\\3\\5\end{bmatrix}, \vec{c_3} = \begin{bmatrix}0\\-1\\-2\end{bmatrix}, \left[\vec{x}\right]_B = \begin{bmatrix}1\\0\\-9\end{bmatrix}$$

$$P_B[x]_B = P_C[x]_C = \vec{x} \qquad P_B = \begin{bmatrix}1&2&1\\1&0&-1\\3&8&3\end{bmatrix} \quad P_C = \begin{bmatrix}2&1&0\\-1&3&-1\\4&5&-2\end{bmatrix}$$

$$\underbrace{P_C^{-1}P_B}_{P_{C\leftarrow B}}[x]_B = [x]_C$$

$$P_{C\leftarrow B} = \begin{bmatrix}1/4&5/4&3/4\\1/2&-1/2&-1/2\\1/4&-11/4&-5/4\end{bmatrix}$$

$$\underbrace{P_B^{-1}P_C}_{P_{B\leftarrow C}}[x]_C = [x]_B$$

$$P_{B\leftarrow C} = \begin{bmatrix}3/2&1&1/2\\-1&1&-1\\5/2&-2&3/2\end{bmatrix}$$

$$P_{C\leftarrow B}\begin{bmatrix}1\\0\\9\end{bmatrix}_B = \begin{bmatrix}1/4&5/4&3/4\\1/2&-1/2&-1/2\\1/4&-11/4&-5/4\end{bmatrix}\begin{bmatrix}1\\0\\9\end{bmatrix} = \begin{bmatrix}-13/2\\5\\23/2\end{bmatrix} = [\vec{x}]_C$$

8. Given the basis $B = \{1-t^2, t-t^2, 2-t+t^2, 2t-t^2+t^3\}$ for P_3. Find $\vec{p}(t) = 2+5t-7t^3$ in this basis. (8 points)

$$\underbrace{\begin{bmatrix}1&0&2&0\\0&1&-1&2\\-1&-1&1&-1\\0&0&0&1\end{bmatrix}^{-1}}_{P_B}\underbrace{\begin{bmatrix}2\\5\\0\\-7\end{bmatrix}}_{p(t)} = [\vec{p}]_B = \begin{bmatrix}-12\\26\\7\\-7\end{bmatrix}$$

9. Prove that $\det A^{-1} = \dfrac{1}{\det A}$ assuming that A is invertible. [Hint: use multiplication properties of the determinant and what you know about $n \times n$ identity matrices.] (5 points)

by def. $A^{-1}A = I$. take determinant of both sides
$\det(A^{-1}A) = \det I = 1$. by properties of determinants, $\det(A^{-1}A)$
$= \det(A^{-1}) \cdot \det A$. Since $\det A$ is a # and if it is not equal
to zero (which must be true if A is invertible) then $\det(A^{-1}) = \dfrac{1}{\det A}$.

10. List at least 8 properties of Invertible Matrices from the Invertible Matrix Theorem. (8 points)

answers will vary but may include:

if A is $n \times n$, then:

1) A reduces to the identity (row equivalent to)
2) A has a pivot in every column
3) A has a pivot in every row
4) $\det A \neq 0$
5) $\dim \operatorname{Nul} A = 0$
6) $\operatorname{rank} A = n$
7) $\dim \operatorname{Col} A = n$
8) columns of A are linearly independent

etc.