MTH 265, Final Exam, Part I, Fall 2018 Name KEY

Instructions: Show all work. Give exact answers unless specifically asked to round. If you do not show work, problems will be graded as "all or nothing" for the answer only; partial credit will not be possible and any credit awarded for the work will not be available.

1. Sketch the gradient field for $z = x^2 + y^2 + x^2y + 4$ and use it to characterize any critical points. (18 points)

$\nabla z = \langle 2x + 2xy,\ 2y + x^2 \rangle$

$2x(1+y) = 0$ $\quad y = -\frac{1}{2}x^2$

$x = 0,\ y = -1$ $\quad -1 = -\frac{1}{2}x^2$

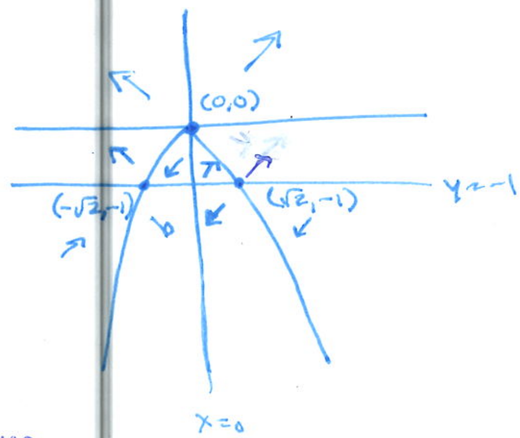
$2x = -2xy$ $\quad 2 = x^2$

$x = \pm\sqrt{2}$

$(2,-3)$ $\langle -4,-2 \rangle$

$(-2,-3)$ $\langle 4, 10 \rangle$

(x,y)	∇z
$(1,1)$	$\langle 4,3 \rangle$
$(-1,1)$	$\langle -4,3 \rangle$
$(-1,-\frac{1}{4})$	$\langle -1, \frac{1}{2} \rangle$
$(1,-\frac{1}{4})$	$\langle 1, \frac{1}{2} \rangle$
$(-\frac{1}{4},-\frac{1}{4})$	$\langle -\frac{1}{4}, -\frac{7}{16} \rangle$
$(\frac{1}{4},-\frac{1}{4})$	$\langle \frac{1}{4}, \frac{9}{16} \rangle$
$(-\frac{1}{4},-2)$	$\langle \frac{1}{2}, -\frac{63}{16} \rangle$
$(\frac{1}{4},-2)$	$\langle -\frac{1}{2}, -\frac{63}{16} \rangle$

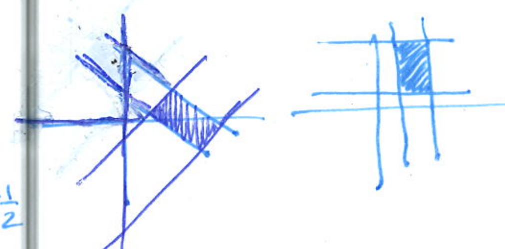


(0,0) minimum

$(\pm\sqrt{2},-1)$ saddle points

2. Find the volume of the region below $f(x,y) = \frac{1}{x^2-y^2}$ for the region bounded by $x+y=1$, $x+y=2$, $x-y=1$, $x-y=4$. (18 points)

$\frac{1}{(x+y)(x-y)}$



$x+y = u$ $\quad [1,2]$

$x-y = v$ $\quad [1,4]$

$2x = u+v$

$x = \frac{1}{2}(u+v)$

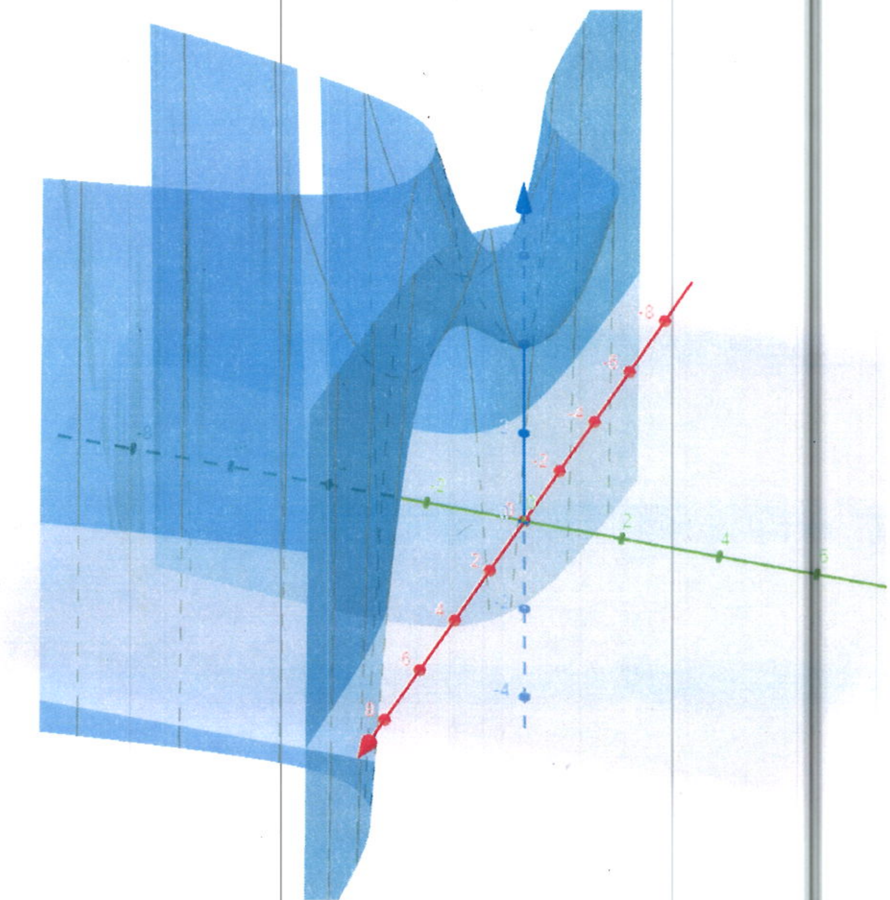
$2y = u-v$

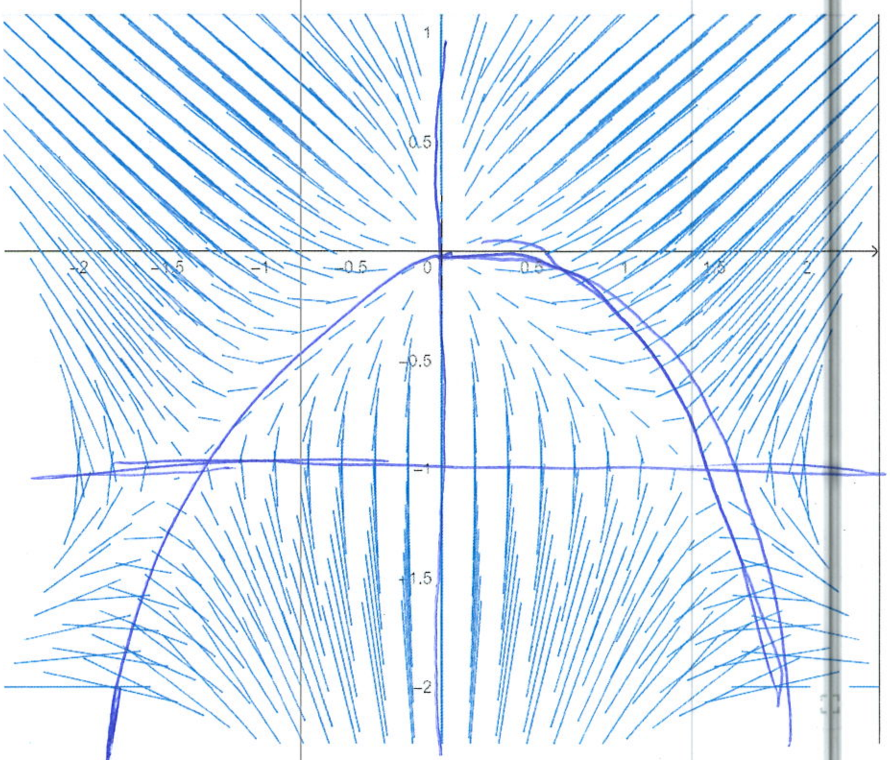
$y = \frac{1}{2}(u-v)$

$J = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2}$

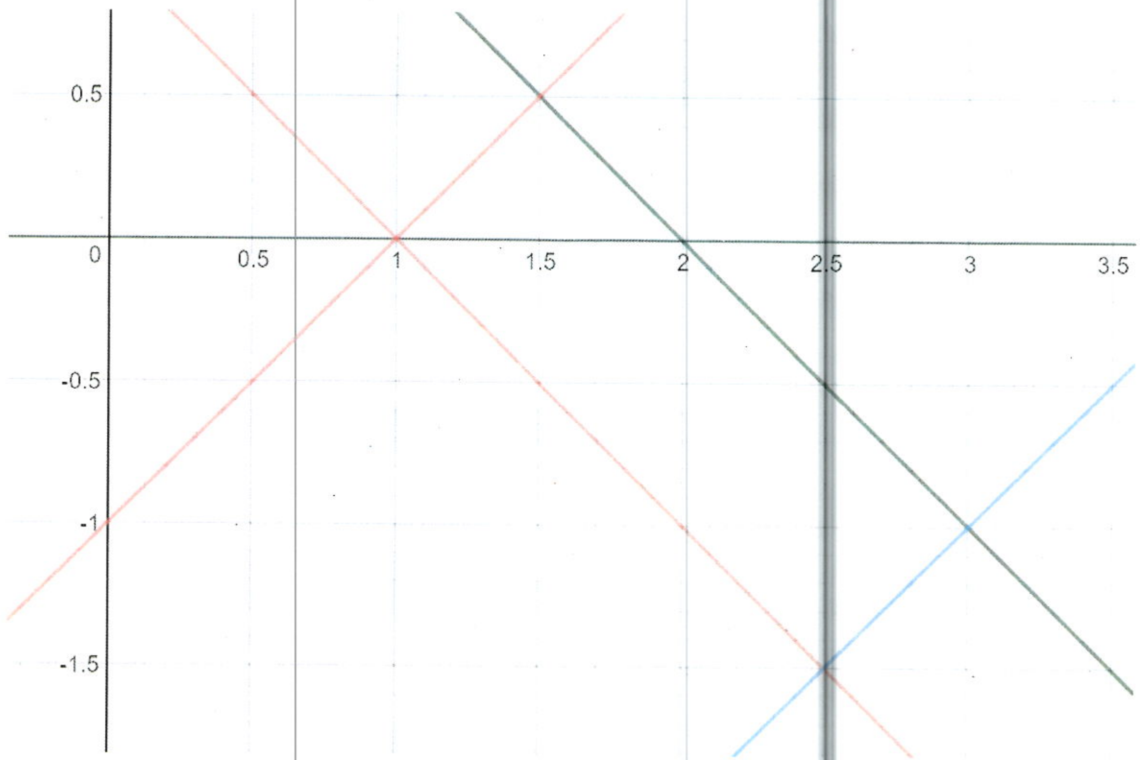
$\int_1^4 \int_1^2 \frac{1}{u} \cdot \frac{1}{v} \left|\frac{1}{2}\right| du\, dv =$

$\frac{1}{2} \int_1^4 \ln u \Big|_1^2 \frac{1}{v} dv = \frac{1}{2}\ln 2 \int_1^4 \frac{1}{v} dv = \frac{1}{2}\ln 2 \cdot \ln 4$





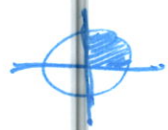
1
0.5
-2
-1.5
-1
-0.5
0
0.5
1.5
2
-0.5
-1
-1.5
-2



0.5
-0.5
-1
-1.5
0
0.5
1
1.5
2
2.5
3
3.5

3. Write the integrals needed to find the center of mass for the region bounded by $0 \le z \le \frac{1}{1+x^2+y^2}, x^2+y^2 \le 4, x \ge 0, y \ge 0, \rho = k\cos\theta$. Find the center of mass. (20 points)

$$M = \int_0^{\pi/2}\int_0^2\int_0^{\frac{1}{1+r^2}} \cos\theta \, r \, dz\, dr\, d\theta$$



$$= \int_0^{\pi/2}\int_0^2 \cos\theta \frac{r}{1+r^2} dr\, d\theta = \int_0^{\pi/2} \frac{1}{2}\ln|1+r^2|\Big|_0^2 \cos\theta\, d\theta =$$

$$= \int_0^{\pi/2}\left[\frac{1}{2}(\ln 5)\right]\cos\theta\, d\theta = \frac{1}{2}\ln 5 \sin\theta\Big|_0^{\pi/2} = \frac{1}{2}\ln 5$$

$$M_{xy} = \int_0^{\pi/2}\int_0^2\int_0^{\frac{1}{1+r^2}} \cos\theta \, z r\, dz\, dr\, d\theta = \int_0^{\pi/2}\int_0^2 \frac{1}{2}z^2\Big|_0^{\frac{1}{1+r^2}} r\cos\theta\, dr\, d\theta$$

$$= \int_0^{\pi/2}\int_0^2 \frac{1}{2}\frac{r}{(1+r^2)^2}\cos\theta\, dr\, d\theta = \int_0^{\pi/2} \frac{-1}{4(r^2+1)}\Big|_0^2 \cos\theta\, d\theta =$$

$$\int_0^{\pi/2} \frac{1}{5}\cos\theta\, d\theta = \frac{1}{5}\sin\theta\Big|_0^{\pi/2} = \frac{1}{5} \qquad \bar{z} = \frac{\frac{1}{5}}{\frac{1}{2}\ln 5} = \frac{2}{5\ln 5}$$

$$M_{xz} = \int_0^{\pi/2}\int_0^2\int_0^{\frac{1}{1+r^2}} r\sin\theta\cos\theta\, r\, dz\, dr\, d\theta = \int_0^{\pi/2}\int_0^2\int_0^{\frac{1}{1+r^2}} r^2 \sin\theta\cos\theta\, dz\, dr\, d\theta$$

$$= \int_0^{\pi/2}\int_0^2 \frac{r^2}{1+r^2}\sin\theta\cos\theta\, dr\, d\theta = \int_0^{\pi/2} \sin\theta\cos\theta(r - \arctan r)\Big|_0^2 d\theta$$

$$= (2-\arctan 2)\int_0^{\pi/2}\sin\theta\cos\theta\, d\theta = \frac{1}{2}(2-\arctan 2)\sin^2\theta\Big|_0^{\pi/2} =$$

$$1 - \frac{1}{2}\arctan 2 \qquad \bar{y} = \frac{\frac{1}{2}(2-\arctan 2)}{\frac{1}{2}\ln 5}$$

$$M_{yz} = \int_0^{\pi/2}\int_0^2\int_0^{\frac{1}{1+r^2}} \cos^2\theta\, r^2\, dz\, dr\, d\theta = \int_0^{\pi/2}\int_0^2 \frac{r^2}{1+r^2}\cos^2\theta\, dr\, d\theta = \frac{2-\arctan 2}{\ln 5}$$

$$= \int_0^{\pi/2} \cos^2\theta(r - \arctan r)\Big|_0^2 d\theta = (2-\arctan 2)\int_0^{\pi/2}\cos^2\theta\, d\theta =$$

$$\frac{1}{2}(2-\arctan 2)\int_0^{\pi/2} 1+\cos 2\theta\, d\theta = \frac{1}{2}(2-\arctan 2)\left(\theta + \frac{1}{2}\sin 2\theta\right)\Big|_0^{\pi/2} =$$

$$\frac{\pi}{4}(2-\arctan 2) \qquad \bar{x} = \frac{\frac{\pi}{4}(2-\arctan 2)}{\frac{1}{2}\ln 5} = \frac{\pi(2-\arctan 2)}{2\ln 5}$$

$$(\bar{x}, \bar{y}, \bar{z}) = \left(\frac{\pi(2-\arctan 2)}{2\ln 5}, \frac{2-\arctan 2}{\ln 5}, \frac{2}{5\ln 5}\right)$$

4. Use the divergence theorem to evaluate $\int_S \int \vec{F} \cdot \vec{N} dS$ for $\vec{F}(x, y, z) = z^2\hat{\imath} + 2x^2\hat{\jmath} + 2xy\hat{k}$ for the closed surface bounded by $x^2 + y^2 = 1, z = 2$ and the coordinate planes. (20 points)

$\vec{\nabla} \cdot \vec{F} = 0$

$\int_0^{\pi/2} \int_0^1 \int_0^2 0 \, dV = 0$

5. Write an equation of the ellipsoid $x^2 + \frac{y^2}{4} + z^2 = 1$ in parametric (surface) form. (10 points)

$r(u, v) = \cos u \sin v \,\hat{\imath} + 2 \sin u \sin v \,\hat{\jmath} + \cos v \,\hat{k}$

MTH 265, Final Exam, Part II, Fall 2018 Name KEY

Instructions: Show all work. Give exact answers unless specifically asked to round. All complex numbers should be stated in standard form, and all complex fractions should be simplified. If you do not show work, problems will be graded as "all or nothing" for the answer only; partial credit will not be possible and any credit awarded for the work will not be available.

6. Find $\frac{dw}{dt}$ for $w = e^{xy} + yz, x = t^2, y = \frac{1}{t}, z = \ln t$ using the chain rule. Be sure your final answer contains only t. (12 points)

$$\frac{\partial w}{\partial x} = ye^{xy} = \frac{1}{t}e^{t} \qquad \frac{dx}{dt} = 2t$$

$$\frac{\partial w}{\partial y} = xe^{xy} + z = t^2e^{t} + \ln t \qquad \frac{dy}{dt} = -\frac{1}{t^2}$$

$$\frac{\partial w}{\partial z} = y = \frac{1}{t} \qquad \frac{dz}{dt} = \frac{1}{t}$$

$$\frac{dw}{dt} = \left(\frac{1}{t}e^{t}\right)(2t) + (t^2e^{t} + \ln t)\left(-\frac{1}{t^2}\right) + \left(\frac{1}{t}\right)\left(\frac{1}{t}\right)$$

$$= 2e^{t} - e^{t} - \frac{\ln t}{t^2} + \frac{1}{t^2} = e^{t} + \frac{1-\ln t}{t^2}$$

7. Find $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ for $x^2y + y^3 + \sqrt{x}\cos x - 5yz = z^2$. (15 points)

$$F = x^2y + y^3 + \sqrt{x}\cos x - 5yz - z^2$$

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{\left(2xy + \frac{1}{2\sqrt{x}}\cos x - \sqrt{x}\sin x\right)}{-5y - 2z}$$

$$= \frac{2xy + \frac{\cos x}{2\sqrt{x}} - \sqrt{x}\sin x}{5y + 2z}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{(x^2 + 3y^2 - 5z)}{-5y - 2z} = \frac{x^2 + 3y^2 - 5z}{5y + 2z}$$

8. Use the second partials test to find any characterize any critical points for $z = 4x + 6y - x^2 - y^2 + xy$. (12 points)

$z_x = 4 - 2x + y = 0 \qquad y = 2x - 4 \qquad 2x - 4 = 3 + \frac{1}{2}x$

$z_y = 6 - 2y + x = 0 \qquad 3 + \frac{1}{2}x = y \qquad 4x - 8 = 6 + x$

$\frac{3x}{3} = \frac{14}{3} \qquad x = \frac{14}{3}$

$y = 3 + \frac{1}{\cancel{2}}\left(\frac{\cancel{14}^{7}}{3}\right) = 3 + \frac{7}{3} = \frac{16}{3}$

critical point $\left(\frac{14}{3}, \frac{16}{3}\right)$

$z_{xx} = -2$

$z_{yy} = -2 \qquad D = (-2)(-2) - (1)^2 = 4 - 1 = 3$ max or min

$z_{xy} = 1$

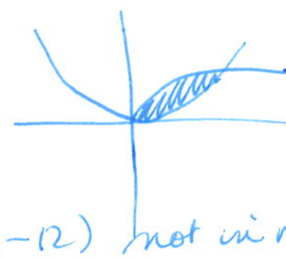
$z_{xx} = -2$ $\cap$ concave down

critical point is a maximum

9. Find the absolute extrema of the function $f(x, y) = x^2 - 3xy + 2y^2 - 6y$ over the region bounded by $y = x^2, and\ y = \sqrt{x}$. (18 points)

$f_x = 2x - 3y = 0 \qquad x = \frac{3}{2}y$

$f_y = -3x + 4y - 6 = 0 \qquad -3\left(\frac{3}{2}y\right) + 4y = 6$



$-\frac{1}{2}y = 6$

$y = -12 \quad ((-18, -12)$ not in region$)$

$x = -18$

$f(x, x^2) = x^2 - 3x(x^2) + 2(x^2)^2 - 6(x^2) = x^2 - 3x^3 + 2x^4 - 6x^2 = 2x^4 - 3x^3 - 5x^2$

$f'(x) = 8x^3 - 9x^2 - 10x = x(8x^2 - 9x - 10) \qquad x = 0 \qquad \underbrace{x = -.68,\ x = 1.81}_{\text{outside region}}$

$f(x, \sqrt{x}) = x^2 - 3x\sqrt{x} + 2(\sqrt{x})^2 - 6\sqrt{x} = x^2 - 3x^{3/2} + 2x - 6x^{1/2}$

$= 2x - \frac{9}{2}x^{1/2} + 2 - \frac{3}{\sqrt{x}} = 0 \qquad x = 4.46$ outside region

corner points $(0,0), (1,1)$

$f(0,0) = 0$ abs max

$f(1,1) = -6$ abs min

10. Find the position vector for $\vec{a}(t) = \frac{1}{t+1}\hat{\imath} - 4\hat{k}, \vec{v}(0) = \hat{\imath} - 3\hat{\jmath} + \hat{k}, \vec{r}(0) = 2\hat{\imath}$. (15 points)

$\int \frac{1}{t+1}\,dt = \ln|t+1| + C_1 = 1 \quad C_1 = 1$

$\int 0\,dt = C_2 = -3 \quad C_2 = -3$

$\int -4\,dt = -4t + C_3 = 1 \quad C_3 = 1$

$v(t) = [\ln|t+1| + 1]\hat{\imath} - 3\hat{\jmath} + (1-4t)\hat{k}$

$\int \ln|t+1| + 1\,dt = (t+1)\ln(t+1) + C_2 = 2 \quad C_2 = 2$

$\int -3\,dt = -3t + C_2 = 0 \quad C_2 = 0$

$\int 1 - 4t\,dt = t - 2t^2 + C_3 = 0 \quad C_3 = 0$

$r(t) = [(t+1)\ln|t+1| + 2]\hat{\imath} - 3t\hat{\jmath} + (t - 2t^2)\hat{k}$

11. Determine the maximum height and range of a projectile fired at height of 0.5 meters above the ground, with an initial velocity of 120 meters/second at an angle of 48° with the horizontal. Use the equation $\vec{r}(t) = (v_0\cos\theta)t\hat{\imath} + \left[h_0 + (v_0\sin\theta)t + \frac{1}{2}gt^2\right]\hat{\jmath}$. (15 points)

120 48° .5 120 48° -9.8

$r(t) = \underbrace{80.3t}_{x(t)}\hat{\imath} + [\underbrace{0.5 + 89.2t - 4.9t^2}_{y(t)}]\hat{\jmath}$

max height

$y'(t) = 0 \qquad 89.2 - 9.8t = 0 \qquad t = 9.1$ secs

$y(9.1) =$ 406.45 meters max height

range

$y(t) = 0 \qquad t = 18.2$

$x(18.2) = 1461.5$ feet = range

12. Find the curvature of the curve $\vec{r}(t) = t^2\hat{\imath} + \cos t\,\hat{\jmath} + \sin t\,\hat{k}$ at the point $t = \frac{\pi}{3}$. What is the radius of curvature at the same point? (14 points)

$r'(t) = 2t\hat{i} + (-\sin t)\hat{j} + \cos t\,\hat{k}$ $\qquad \|r'(t)\| = \sqrt{4t^2+\cos^2 t+\sin^2 t}$

$r''(t) = 2\hat{i} - \cos t\,\hat{j} - \sin t\,\hat{k}$ $\qquad = \sqrt{4t^2+1}$

$@\frac{\pi}{3} \quad = \sqrt{\frac{4\pi^2}{9}+1}$

$$r' \times r'' = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2t & -\sin t & \cos t \\ 2 & -\cos t & -\sin t \end{vmatrix} = (\sin^2 t + \cos^2 t)\hat{i} - (-2t\sin t - 2\cos t)\hat{j} + (-2t\cos t + 2\sin t)\hat{k}$$

$= \hat{i} + (2t\sin t + 2\cos t)\hat{j} + (2\sin t - 2t\cos t)\hat{k}$

$@\frac{\pi}{3} = \hat{i} + \left(\frac{\pi\sqrt{3}}{3}+1\right)\hat{j} + \left(\sqrt{3} - \pi/3\right)\hat{k}$

$$K = \frac{\sqrt{1+\left(\frac{\pi\sqrt{3}}{3}+1\right)^2+\left(\sqrt{3}-\pi/3\right)^2}}{\left(\frac{4\pi^2}{9}+1\right)^{3/2}}$$

$$= \sqrt{\frac{4\pi^2/9+5}{(4\pi^2/9+1)^3}} \approx 0.245$$

$$R = \frac{1}{K} = \sqrt{\frac{(4\pi^2/9+1)^3}{4\pi^2/9+5}} \approx 4.08$$

13. Find the equation of the tangent plane to the surface $x^2z^2 = y^2$ at $(1,4,-2)$. (12 points)

$F = x^2z^2 - y^2 = 0$

$\nabla F = \langle 2xz^2, -2y, 2x^2z \rangle$

$\nabla F(1,4,-2) = \langle 8, -8, -4 \rangle$

plane:

$8(x-1) - 8(y-4) - 4(z+2) = 0$

14. Use Green's Theorem to evaluate $\int_C (\overset{M}{y - e^x})dx + (\overset{N}{2x - \ln y})dy$ on the path described by the boundary of circle $x^2 + y^2 = 9$ oriented counterclockwise. (15 points)

$$\frac{\partial N}{\partial x} = 2 \qquad \frac{\partial M}{\partial y} = 1$$

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 2 - 1 = 1$$

$$\int_0^{2\pi}\int_0^3 1 \; r\,dr\,d\theta = \text{Area of Circle} * 1 = 9\pi$$

15. Find the limit $\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^6+y^3}$. (10 points)

$$x^6 = y^3$$
$$kx^2 = y$$

$$\lim_{x\to 0} \frac{x^2 \cdot kx^2}{x^6 + k^3x^6} = \lim_{x\to 0} \frac{x^4(k)}{x^6(1+k^3)} = \lim_{x\to 0} \frac{k}{x^2(1+k^3)} = \text{DNE}$$

undefined and there can be

a sign flip while $\frac{1}{x^2}$ goes to $+\infty$

if $k = 1$ lim $\to +\infty$

but if $k = -1$ lim $\to -\infty$

16. Convert the triple integral $\int_{-7}^{7}\int_0^{\sqrt{49-x^2}}\int_{x^2+y^2}^{\sqrt{49-x^2-y^2}} \frac{\ln\sqrt{x^2+y^2+z^2}+1}{x^2+y^2+z^2}\,dzdydx$ to spherical coordinates and then complete the integration. Describe the region being integrated (over). (15 points)

$\int_0^{\pi}\int_0^{.3733}\int_0^{7} \frac{\ln(\rho+1)}{\cancel{\rho^2}}\cancel{\rho^2}\sin\varphi\, d\rho\, d\varphi\, d\theta +$

$+\int_0^{\pi}\int_{.3733}^{\pi/2}\int_0^{\cot\varphi\csc\varphi} \frac{\ln(\rho+1)}{\cancel{\rho^2}}\cancel{\rho^2}\sin\varphi\, d\rho\, d\varphi\, d\theta =$

$\int_0^{\pi}\int_0^{0.3733} (8\ln 8-8)\sin\varphi\, d\varphi\, d\theta +$

$\int_0^{\pi}\int_{0.3733}^{\pi/2} \cot\varphi\,\cancel{\csc\varphi}\,\ln|\csc\varphi\cot\varphi|\,\cancel{\sin\varphi}\, d\varphi\, d\theta =$

$\int_0^{\pi} 0.594738\, d\theta + \int_0^{\pi} 0.640683\, d\theta =$

$= 1.94059$

+y region of volume bounded by paraboloid and top half of sphere

$\rho\cos\varphi = \rho^2\sin^2\varphi$

$\frac{\cos\varphi}{\sin\varphi}\cdot\frac{1}{\sin\varphi} = \rho$

$\rho = \cot\varphi\csc\varphi$

$x^2+y^2 = \sqrt{49-x^2-y^2}$

$r^2 = \sqrt{49-r^2}$

$r^4 = 49-r^2$

$r^4+r^2-49=0$

$r^2 = 6.5178344$

$r = 2.55$

$r^2 = \rho^2\sin^2\varphi$

$\frac{r^2}{\rho^2} = 0.133 = \sin^2\varphi$

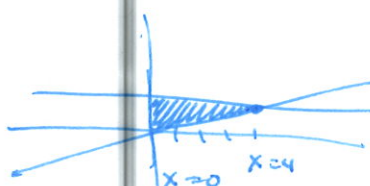
$\varphi = 21.39^\circ$ or $.3733$ rads

17. Change the order of integration in $\int_0^4\int_{x/4}^1 \frac{1}{1+y^4}\,dydx$ so that it can be integrated. Then complete the integration. (12 points)

$y = \frac{x}{4}$

$4y = x$

$x = 0$



$\int_0^1\int_0^{4y} \frac{1}{1+y^4}\,dx\,dy$

$\int_0^1 \frac{4y}{1+y^4}\,dy$

$u = y^2$

$du = 2y\,dy$

$2\int \frac{du}{1+u^2}$

$2\arctan y^2 \Big|_0^1$

$= 2\arctan 1 = 2\cdot\frac{\pi}{4} = \frac{\pi}{2}$

18. Determine if the vector field $\vec{F}(x,y,z) = (\cos y + 3x^2)\hat{\imath} - (x \sin y + z)\hat{\jmath} - y\hat{k}$ is conservative. Find the potential function. (12 points)

$$\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \cos y + 3x^2 & -x\sin y - z & -y \end{vmatrix} = (-1+1)\hat{\imath} - (0-0)\hat{\jmath} + (-\sin y + \sin y)\hat{k}$$

$= \vec{0}$ Conservative

$\int \cos y + 3x^2 \, dx = x\cos y + x^3 + f(y,z)$

$\int -x\sin y - z \, dy = x\cos y - yz + g(x,z)$

$\int -y \, dz = -yz + h(x,y)$

$\varphi = x\cos y + x^3 - yz + K$

19. Evaluate the line integral $\int_C y(x+z)ds$ on the path $\vec{r}(t) = \sqrt{t}\hat{\imath} + t^2\hat{\jmath} + 2t\hat{k}$ on $[0,1]$. (12 points)

$r'(t) = \frac{1}{2\sqrt{t}} + 2t\hat{\jmath} + 2\hat{k}$

$\|r'(t)\| = \sqrt{\frac{1}{4t} + 4t^2 + 4}$

$\int_0^1 t^2(\sqrt{t} + 2t)\sqrt{\frac{1}{4t} + 4t^2 + 4}\, dt \approx 2.069$

Cylindrical

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z$$
$$x^2 + y^2 = r^2$$
$$\tan^{-1}\left(\frac{y}{x}\right) = \theta$$

Spherical

$$x = \rho\cos\theta\sin\phi$$
$$y = \rho\sin\theta\sin\phi$$
$$z = \rho\cos\phi$$
$$x^2 + y^2 + z^2 = \rho^2$$
$$\tan^{-1}\left(\frac{y}{x}\right) = \theta$$
$$\cos^{-1}\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right) = \phi$$
$$x^2 + y^2 = \rho^2\sin^2\phi = r^2$$

Dels

$$\frac{\partial}{\partial x} = \text{ partial derivative with respect to } x$$
$$\nabla f = grad\ f$$
$$\nabla^2 f = \nabla\cdot(\nabla f) = Laplacian\ of\ f$$
$$\nabla\cdot\vec{F} = div\ \vec{F}$$
$$\nabla\times\vec{F} = curl\ \vec{F}$$

Misc

$$ds = \|\vec{r}'(t)\|dt$$
$$\kappa = \frac{\|r' \times r''\|}{\|r'\|^3} = \frac{1}{R}$$
$$\vec{N} = \nabla F = \vec{r}_u \times \vec{r}_v$$
$$D = f_{xx}f_{yy} - \left(f_{xy}\right)^2$$