MTH 265, Exam #3, Part I, Fall 2018 Name KEY

Instructions: Show all work. Give exact answers unless specifically asked to round. If you do not show work, problems will be graded as "all or nothing" for the answer only; partial credit will not be possible and any credit awarded for the work will not be available.

1. Find the curvature of the curve $\vec{r}(t) = t^3\hat{\imath} + \ln t\,\hat{\jmath} + \tan^{-1} t\,\hat{k}$ at the point $t = 1$. What is the radius of curvature at the same point? (12 points)

$r' = 3t^2\hat{\imath} + \frac{1}{t}\hat{\jmath} + \frac{1}{1+t^2}\hat{k}$ $\qquad r'(1) = 3\hat{\imath} + 1\hat{\jmath} + \frac{1}{2}\hat{k}$

$r'' = 6t\hat{\imath} - \frac{1}{t^2}\hat{\jmath} + \frac{-2t}{(1+t^2)^2}\hat{k}$ $\qquad r''(1) = 6\hat{\imath} - 1\hat{\jmath} - \frac{1}{2}\hat{k}$

$$r' \times r'' = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 3 & 1 & 1/2 \\ 6 & -1 & -1/2 \end{vmatrix} = \underbrace{\left(-\tfrac{1}{2}+\tfrac{1}{2}\right)}_{0}\hat{\imath} - \underbrace{\left(-\tfrac{3}{2}-3\right)}_{+\frac{9}{2}}\hat{\jmath} + \underbrace{(-3-6)}_{-9}\hat{k}$$

$$\frac{\|r' \times r''\|}{\|r'\|^3} = \frac{\sqrt{\frac{81}{4}+81}}{\left(\sqrt{9+1+\frac{1}{4}}\right)^3} = \frac{\sqrt{\frac{405}{4}}}{\frac{41\sqrt{41}}{4\sqrt{4}}} = \frac{4\sqrt{405}}{41\sqrt{41}} = K(1)$$

$$R(1) = \frac{1}{K} = \frac{41\sqrt{41}}{4\sqrt{405}}$$

2. Find an equation of the tangent plane to the surface $\vec{r}(u,v) = u\cos v\,\hat{\imath} + u\sin v\,\hat{\jmath} + u\hat{k}$ at the point $u = 3, v = \frac{\pi}{4}$. (8 points)

$\left(\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}, 3\right)$

$$r_u \times r_v = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \cos v & \sin v & 1 \\ -u\sin v & u\cos v & 0 \end{vmatrix} = (0 - u\cos v)\hat{\imath} - (0 + u\sin v)\hat{\jmath} + (u\cos^2 v + u\sin^2 v)\hat{k}$$

$-u\cos v\,\hat{\imath} - u\sin v\,\hat{\jmath} + u\hat{k}$

$\left\langle -\frac{3}{\sqrt{2}}, -\frac{3}{\sqrt{2}}, 3 \right\rangle$

$$-\frac{3}{\sqrt{2}}\left(x - \frac{3}{\sqrt{2}}\right) - \frac{3}{\sqrt{2}}\left(y - \frac{3}{\sqrt{2}}\right) + 3(z-3) = 0$$

3. Find the area of the surface given by the part of the sphere $x^2 + y^2 + z^2 = 4z$ that lies inside the paraboloid $z = x^2 + y^2$. (8 points)

$x^2+y^2+(x^2+y^2)^2 = 4(x^2+y^2)$

$r^2 + r^4 = 4r^2$

$r^4 - 3r^2 = 0$

$r^2(r^2-3)=0$

$r=0$

$r=\sqrt{3}$

$\int_0^{2\pi}\int_0^{\sqrt{3}} \frac{2r}{\sqrt{4-r^2}}\, dr\, d\theta$

$u=4-r^2$

$du=-2r\,dr$

$\int u^{-1/2}\,du$

$-2u^{1/2}$

$\int_0^{2\pi} -2\sqrt{4-r^2}\Big|_0^{\sqrt{3}} d\theta$

$\cancel{-2}\int_0^{2\pi} 1-2\, d\theta = 2\pi \cdot 2 = 4\pi$

$x^2+y^2+(z-2)^2=4$

$(z-2)^2 = 4-x^2-y^2$

$z-2 = \sqrt{4-x^2-y^2}$

$z = 2+\sqrt{4-x^2-y^2}$

$G = 2-z+\sqrt{4-x^2-y^2}$

$\nabla G = \left\langle \frac{-x}{\sqrt{4-x^2-y^2}}, \frac{-y}{\sqrt{4-x^2-y^2}}, -1 \right\rangle$

$\|\nabla G\| = \sqrt{\frac{x^2}{4-x^2-y^2}+\frac{y^2}{4-x^2-y^2}+1}$

$= \sqrt{\frac{\cancel{4-x^2-y^2+x^2+y^2}}{4-x^2-y^2}}$

$= \frac{2}{\sqrt{4-x^2-y^2}} = \frac{2}{\sqrt{4-r^2}}$

4. Evaluate the flux integral $\int_S \int \vec{F} \cdot \vec{N} dS$ where $\vec{F} = 2z\hat{\imath} - 4x\hat{\jmath} + \hat{k}$ for the surface $S: z = 12 - 3x - 4y$ in the first octant. (12 points)

$G=12-3x-4y-z$

$\langle -3,-4,-1\rangle = \nabla G$ downward or $\langle 3,4,1\rangle$ upward

$F\cdot\nabla G = \langle 2z,-4x,1\rangle \cdot \langle 3,4,1\rangle = 6z-16x+1$

$= 6(12-3x-4y)-16x+1 = 72-18x-24y-16x+1 = 73-34x-24y$

$\int_0^4\int_0^{3-3/4x} 73-34x-24y\, dy\, dx$

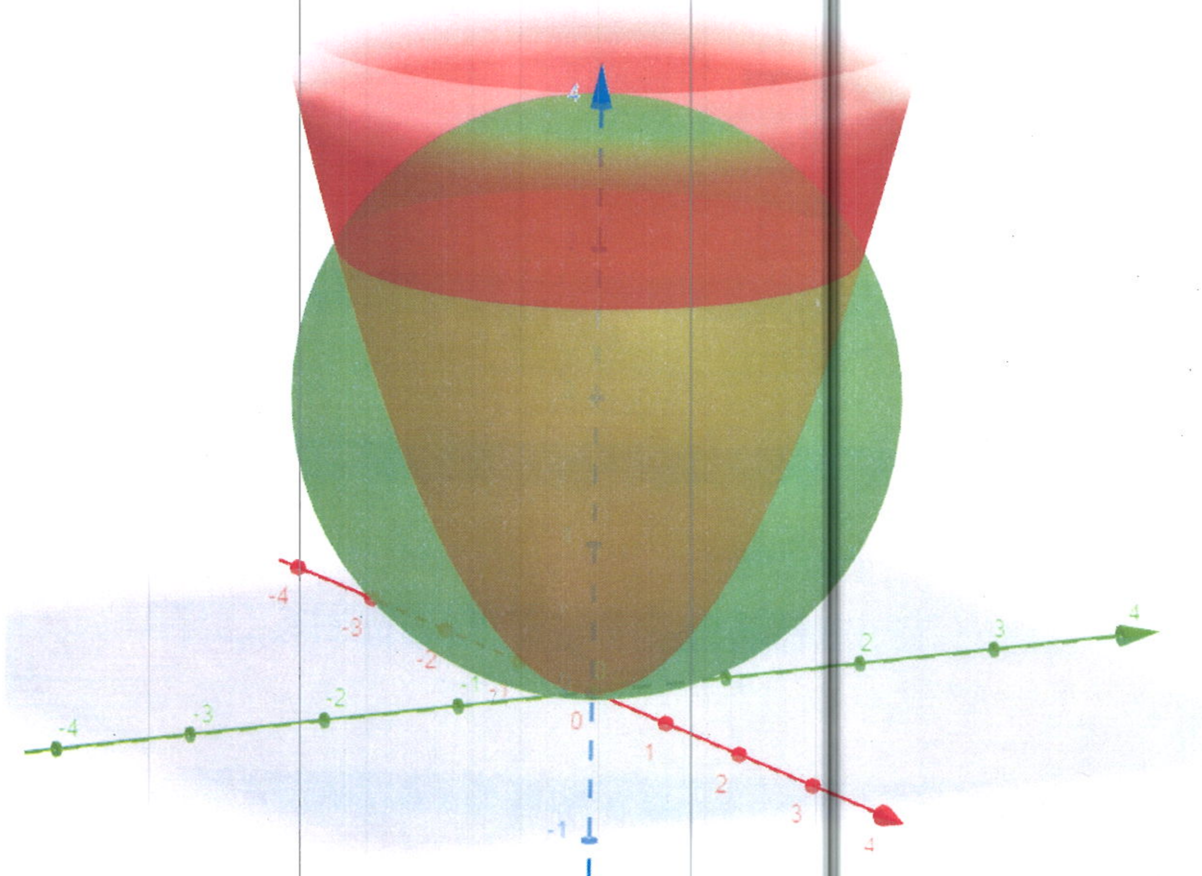
$\frac{12=3x+4y}{4}$

$3 = \frac{3}{4}x+y$

$3-\frac{3}{4}x = y$

$\int_0^4 73y-34xy-12y^2\Big|_0^{3-3/4x} dx =$

$\int_0^4 \frac{75}{4}x^2 - \frac{411}{4}x + 111\, dx = \frac{25}{4}x^3 - \frac{411}{8}x^2 + 111x\Big|_0^4 = 22$



4
-4
-3
-2
-1
0
1
2
3
4
-4
-3
-2
-1
1
2
3
4
-1

5. Use the divergence theorem to evaluate $\int_S \int \vec{F} \cdot \vec{N} dS$ for $\vec{F}(x,y,z) = (2xy-1)\hat{\imath} + (3yz+2)\hat{\jmath} + xz\hat{k}$ for the closed surface bounded by the cylinder $x^2+y^2=4, z=4$, and the coordinate planes. (14 points)

$$\vec{\nabla}\cdot\vec{F} = 2y + 3z + x = 2r\sin\theta + 3z + r\cos\theta$$

$$\int_0^{\pi/2}\int_0^2\int_0^4 2r^2\sin\theta + 3rz + r^2\cos\theta \, dz\,dr\,d\theta =$$

$$\int_0^{\pi/2}\int_0^2 2r^2\sin\theta z + \frac{3}{2}rz^2 + r^2 z\cos\theta \Big|_0^4 dr\,d\theta =$$

$$\int_0^{\pi/2}\int_0^2 8r^2\sin\theta + 24r + 4r^2\cos\theta \, dr\,d\theta =$$

$$\int_0^{\pi/2} \frac{8}{3}r^3\sin\theta + 12r^2 + \frac{4}{3}r^3\cos\theta \Big|_0^2 d\theta =$$

$$\int_0^{\pi/2} \frac{64}{3}\sin\theta + 48 + \frac{32}{3}\cos\theta \, d\theta = -\frac{64}{3}\cos\theta + 48\theta + \frac{32}{3}\sin\theta \Big|_0^{\pi/2}$$

$$= 24\pi + 32$$

6. Evaluate $\int_C \vec{F}\cdot d\vec{r}$ using Stokes' Theorem for $\vec{F}(x,y,z) = z\hat{\imath} + x\hat{\jmath} + xyz\hat{k}$ for $S: z = x^2 - y^2$, $0 \le x \le 1, 0 \le y \le 1$. (12 points)

$$\nabla\times F = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ z & x & xyz \end{vmatrix} = (xz-0)\hat{\imath} - (yz-1)\hat{\jmath} + (1-0)\hat{k}$$

$$\langle xz, 1-yz, 1\rangle$$

$$G = -x^2 + y^2 + z \qquad \nabla G = \langle -2x, 2y, 1\rangle$$

$$(\nabla\times F)\cdot\nabla G = -2x^2z + 2y - 2y^2z + 1 = -2x^2(x^2-y^2) - 2y^2(x^2-y^2) + 1 + 2y$$

$$= -2x^4 + \cancel{2x^2y^2} - \cancel{2x^2y^2} + 2y^4 + 1 + 2y$$

$$\int_0^1\int_0^1 -2x^4 + 2y^4 + 1 + 2y \, dy\,dx = \int_0^1 \frac{2}{5}y^5 + y - 2x^4y \Big|_0^1 dx = \int_0^1 \frac{12}{5} - 2x^4 \, dx$$

$$+y^2$$

$$= \frac{12}{5}x - \frac{2}{5}x^5\Big|_0^1 = 2$$

MTH 265, Exam #3, Part II, Fall 2018 Name KEY

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7. Find the unit tangent vector for $\vec{r}(t) = (\sqrt{t^3} - 4)\hat{\imath} + (t^2 + 1)\hat{\jmath}$. (8 points)

$t^{3/2}$

$\vec{r}'(t) = \left(\frac{3}{2}t^{1/2}\right)\hat{\imath} + 2t\hat{\jmath}$

$\|\vec{r}'(t)\| = \sqrt{\frac{9}{4}t + 4t^2}$

$$\vec{T}(t) = \frac{\frac{3\sqrt{t}}{2}\hat{\imath} + 2t\hat{\jmath}}{\sqrt{\frac{9}{4}t + 4t^2}}$$

8. Write an integral for the arc length of the curve $\vec{r}(t) = \sqrt[3]{t^4}\hat{\imath} + \sqrt{t^5}\hat{\jmath} + t\hat{k}$ on the interval $[0,2]$. Evaluate it numerically. (6 points)

$t^{4/3}$ $t^{5/2}$

$r'(t) = \frac{4}{3}t^{1/3}\hat{\imath} + \frac{5}{2}t^{3/2}\hat{\jmath} + 1\hat{k}$

$\|r'(t)\| = \sqrt{\frac{16}{9}t^{2/3} + \frac{25}{4}t^3 + 1}$

$$\int_0^2 \sqrt{\frac{16}{9}t^{2/3} + \frac{25}{4}t^3 + 1}\,dt \approx 6.1378$$

6.84028

9. Find the directional derivative of the function $w = \arcsin xyz$ at the point $\left(1, \frac{1}{2}, 1\right)$ in the direction of $\vec{v} = \langle -1, 4, -2 \rangle$. In what direction is the directional derivative a maximum? (10 points)

$$\nabla w = \left\langle \frac{yz}{\sqrt{1-x^2y^2z^2}}, \frac{xz}{\sqrt{1-x^2y^2z^2}}, \frac{xy}{\sqrt{1-x^2y^2z^2}} \right\rangle$$

$$\hat{v} = \left\langle \frac{-1}{\sqrt{21}}, \frac{4}{\sqrt{21}}, \frac{-2}{\sqrt{21}} \right\rangle$$

$$\nabla w\left(1, \tfrac{1}{2}, 1\right) = \left\langle \frac{\frac{1}{2}}{\sqrt{1-\frac{1}{4}}}, \frac{1}{\sqrt{1-\frac{1}{4}}}, \frac{\frac{1}{2}}{\sqrt{1-\frac{1}{4}}} \right\rangle$$

$$= \left\langle \frac{1}{\cancel{2}} \cdot \sqrt{\frac{\cancel{4}}{3}}, 1 \cdot \sqrt{\frac{4}{3}}, \frac{1}{\cancel{2}} \cdot \sqrt{\frac{\cancel{4}}{3}} \right\rangle =$$

$$\left\langle \frac{1}{\sqrt{3}}, \frac{2}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle$$

Max

$= \nabla w$

$= \left\langle \frac{1}{\sqrt{3}}, \frac{2}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle$

$$\nabla w \cdot \hat{v} = \left\langle \frac{1}{\sqrt{3}}, \frac{2}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle \cdot \left\langle \frac{-1}{\sqrt{21}}, \frac{4}{\sqrt{21}}, \frac{-2}{\sqrt{21}} \right\rangle =$$

$$\frac{-1}{\sqrt{63}} + \frac{8}{\sqrt{63}} - \frac{2}{\sqrt{63}} = \frac{5}{\sqrt{63}} = \frac{5}{3\sqrt{7}}$$

10. Find the equation of the tangent plane to the surface $xyz + z^{2/3} = y^2$ at $(-1,1,1)$. (8 points)

$$F = xyz + z^{2/3} - y^2$$

$$\nabla F = \left\langle yz, xz - 2y, xy + \tfrac{2}{3} z^{-1/3} \right\rangle$$

$$\nabla F(-1,1,1) = \left\langle 1, -1-2, -1 + \tfrac{2}{3} \right\rangle = \left\langle 1, -3, -\tfrac{1}{3} \right\rangle$$

plane:

$$1(x+1) - 3(y-1) - \tfrac{1}{3}(z-1) = 0$$

11. Use the Fundamental Theorem of Line Integrals to evaluate $\int_C (2xy+1)dx + (x^2 - y)dy$ on the line segment from $(0,2)$ to $(4,-2)$. (10 points)

φ : $\int 2xy + 1\, dx = x^2y + x + f(y)$

$\int x^2 - y\, dy = x^2y - \frac{1}{2}y^2 + g(x)$

$\varphi = x^2y + x - \frac{1}{2}y^2$

$\varphi(4,-2) - \varphi(0,2) = 16(-2) + 4 - \frac{1}{2}(4) - [0 + 0 - \frac{1}{2}(4)] =$

$-32 + 4 - \cancel{2} + \cancel{2} = -28$

12. Use Green's Theorem to evaluate $\int_C \overset{M}{xy}dx + \overset{N}{x^2}dy$ on the path described by the boundary of the graphs $y = x^2, y = \sqrt{x}$ oriented counterclockwise. (12 points)

$\frac{\partial M}{\partial y} = x \qquad \frac{\partial N}{\partial x} = 2x$

$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 2x - x = x$

$\int_0^1 \int_{x^2}^{\sqrt{x}} x\, dy\, dx = \int_0^1 xy \Big|_{x^2}^{\sqrt{x}} dx = \int_0^1 x^{3/2} - x^3\, dx$

$= \frac{2}{5}x^{5/2} - \frac{1}{4}x^4 \Big|_0^1 = \frac{2}{5} - \frac{1}{4} = \frac{3}{20}$

13. Evaluate the surface integral $\int_S \int f(x,y,z)dS$ for $f(x,y,z) = x^2z^2$, S: on the cone $z^2 = x^2 + y^2$, between the planes $z = 1, z = 3$. [Hint: converting to cylindrical/polar will help.] (12 points)

$Z = r$

$Z = \sqrt{x^2+y^2}$

$G = Z - \sqrt{x^2+y^2}$

$\nabla G = \left\langle \frac{-x}{\sqrt{x^2+y^2}}, \frac{-y}{\sqrt{x^2+y^2}}, 1 \right\rangle$

$\|\nabla G\| = \sqrt{\frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2} + 1\frac{(x^2+y^2)}{(x^2+y^2)}} = \sqrt{\frac{2x^2+2y^2}{x^2+y^2}} = \sqrt{2}$

$\int_0^{2\pi}\int_1^3 \sqrt{2}\, r^2 \cos^2\theta \, z^2 r \, dr\, d\theta = \int_0^{2\pi}\int_1^3 \sqrt{2}\, r^5 \cos^2\theta \, dr\, d\theta$

$\uparrow$

$x^2+y^2=r^2$

$\int_0^{2\pi} \frac{\sqrt{2}}{6} r^6 \Big|_1^3 \cos^2\theta \, d\theta = \int_0^{2} \frac{364\sqrt{2}}{3} \cos^2\theta \, d\theta$

$= \int_0^{2\pi} \frac{182\sqrt{2}}{3} (1 + \cos 2\theta)\, d\theta = \frac{182\sqrt{2}}{3}\left[\theta + \frac{1}{2}\overset{0}{\cancel{\sin 2\theta}}\right]_0^{2\pi}$

$= \frac{364\sqrt{2}}{3}\pi$

Cylindrical

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z$$
$$x^2 + y^2 = r^2$$
$$\tan^{-1}\left(\frac{y}{x}\right) = \theta$$

Spherical

$$x = \rho\cos\theta\sin\phi$$
$$y = \rho\sin\theta\sin\phi$$
$$z = \rho\cos\phi$$
$$x^2 + y^2 + z^2 = \rho^2$$
$$\tan^{-1}\left(\frac{y}{x}\right) = \theta$$
$$\cos^{-1}\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right) = \phi$$
$$x^2 + y^2 = \rho^2\sin^2\phi = r^2$$

Dels

$$\frac{\partial}{\partial x} = \text{ partial derivative with respect to } x$$
$$\nabla f = grad\ f$$
$$\nabla^2 f = \nabla\cdot(\nabla f) = Laplacian\ of\ f$$
$$\nabla\cdot\vec{F} = div\ \vec{F}$$
$$\nabla\times\vec{F} = curl\ \vec{F}$$

Misc

$$ds = \|\vec{r}'(t)\|dt$$
$$\kappa = \frac{\|r'\times r''\|}{\|r'\|^3} = \frac{1}{R}$$
$$\vec{N} = \nabla F = \vec{r}_u\times\vec{r}_v$$