MTH 291 Skills #6 Key



1a. $R=\sqrt{3^2+4^2}=5$ $\qquad \omega=2$

$\delta = \tan^{-1}\left(\frac{4}{3}\right) \approx .9273$

$u(t) = 5\cos(2t - .9273)$

b. $R=\sqrt{(-2)^2+(-3)^2}=\sqrt{13}$ $\qquad \omega=\pi$

$\delta = \tan^{-1}\left(\frac{+3}{+2}\right) \approx .9828$

$u(t) = \sqrt{13}\cos(\pi t - .9828)$

2. $Lq'' + Rq' + \frac{1}{C}q = 0$

$q' = i$

$q'' = -\frac{R}{L}i - \frac{1}{CL}q$

$\frac{dq}{dt} = i$

$\frac{di}{dt} = -\frac{R}{L}i - \frac{1}{CL}q$ System

$q = e^{rt} \Rightarrow Lr^2 + Rr + \frac{1}{C} = 0$

$0.2r^2 + Rr + \frac{10^6}{.8} = 0$

$b^2 - 4ac = 0$

$R^2 = 4(0.2)\left(\frac{10^6}{.8}\right) = 10^6$

$R = 1000\,\Omega$

3. $q(0) = 10^{-6}$ $\quad q'(0) = 0$

$0.2q'' + 300 + 10^5 = 0$

$r = \frac{-300 \pm \sqrt{300^2 - 4(.2)10^5}}{2(.2)} = \frac{-300 \pm 100}{.4}$

$= \frac{-200}{.4} = -500 \qquad \frac{-400}{.4} = -1000$

$A(e^{-500t}) + B(e^{-1000t}) = q(t)$

$A + B = 10^{-6}$

$-500A - 1000B = 0$

$\frac{-1000B}{500} = A \Rightarrow -2B = A \Rightarrow A = 2\times10^{-6}$

$-2B + B = 10^{-6} \Rightarrow B^{-6}$

$q(t) = 2\times10^{-6}e^{-500t} + 10^{-6}e^{-1000t}$

4. $mx'' + \gamma x' + kx = 0 \Rightarrow$

$\frac{dx}{dt} = y$

$\frac{dy}{dt} = -\frac{k}{m}x - \frac{\gamma}{m}y$

$k=18,\ m=2\ \ \gamma=4$

$2x'' + 4x' + 18x = 0$

$\frac{dx}{dt} = y$

$\frac{dy}{dt} = -9x - 2y$

4 cont'd

$x'' + 2x' + 9x = 0$

$r^2 + 2r + 9 = 0$

$x(0) = 1,\ x'(0) = 0$

$r = \frac{-2 \pm \sqrt{4-4(9)}}{2} = \frac{-2 \pm \sqrt{-32}}{2} = \frac{-2 \pm 4\sqrt{2}i}{2}$

$= -1 \pm 2\sqrt{2}i$

$x = Ae^{-t}\cos(2\sqrt{2}t) + Be^{-t}\sin(2\sqrt{2}t) \Rightarrow A + \cancel{B(0)} = 1$

$x' = -Ae^{-t}\cos(2\sqrt{2}t) - \underset{\nearrow 0}{2\sqrt{2}Ae^{-t}\sin(2\sqrt{2}t)} - \underset{\nearrow 0}{Be^{-t}\sin(2\sqrt{2}t)} + 2\sqrt{2}Be^{-t}\cos(2\sqrt{2}t)$

$A = 1$

$0 = -\cancel{e^{-t}}\cancel{\cos(2\sqrt{2}t)} + 2\sqrt{2}B\cancel{e^{-t}}\cancel{(\cos 2\sqrt{2}t)}$

$1 = 2\sqrt{2}B \Rightarrow B = \frac{1}{2\sqrt{2}}$

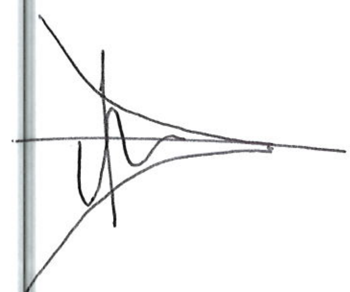
a. $x(t) = e^{-t}\cos(2\sqrt{2}t) + \frac{1}{2\sqrt{2}}e^{-t}\sin(2\sqrt{2}t)$

Underdamped

b. $R = \sqrt{1 + \left(\frac{1}{2\sqrt{2}}\right)^2} = \sqrt{1 + \frac{1}{8}} = \sqrt{\frac{9}{8}} = \frac{3}{2\sqrt{2}}$

$\delta = \tan^{-1}\left(\frac{\frac{1}{2\sqrt{2}}}{1}\right) \approx .3398$

$x(t) = \frac{3}{2\sqrt{2}}\cos(2\sqrt{2}t + .3398)e^{-t}$



5. a. underdampeded
   b. undamped
   c. over damped/critically damped.

6. a. beats $x'' + 81x = \cos 10t$
   b. resonance $x'' + x = \cos t$
   c. $x'' + 7x' + 12x = 0$
   d. can reuse #c
   e. $x'' + 7x' + 12x = \sin 2t$
   f. $x'' + x = 0$
   g. $x'' + 6x' + 9x = 0$

answers will vary

7. $\vec{x} = \begin{pmatrix}4\\2\end{pmatrix}e^{2t}$ $x' = 2\begin{pmatrix}4\\2\end{pmatrix}e^{2t} = \begin{pmatrix}8\\4\end{pmatrix}e^{2t}$

$\begin{pmatrix}3 & -2\\2 & -2\end{pmatrix}\begin{pmatrix}4\\2\end{pmatrix}e^{2t} = \begin{pmatrix}8\\4\end{pmatrix}e^{2t}$ ✓

8. $X = \begin{pmatrix} 6 \\ -8 \\ -4 \end{pmatrix} e^{-t} + 2\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} e^{2t}$

$X' = \begin{pmatrix} -6 \\ 8 \\ 4 \end{pmatrix} e^{-t} + 4\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} e^{2t}$

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} \vec{X} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} 6 \\ -8 \\ -4 \end{pmatrix} e^{-t} + 2\begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} e^{2t} = \begin{pmatrix} -6 \\ 8 \\ 4 \end{pmatrix} e^{-t} + \begin{pmatrix} 0 \\ 4 \\ -4 \end{pmatrix} e^{2t}$$

✓

9. $\Psi' = \begin{pmatrix} -3e^{-3t} & 2e^{2t} \\ 12e^{-3t} & 2e^{2t} \end{pmatrix}$

$$\begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} e^{-3t} & e^{2t} \\ -4e^{-3t} & e^{2t} \end{pmatrix} = \begin{pmatrix} e^{-3t} - 4e^{-3t} & e^{2t} + e^{2t} \\ 4e^{-3t} + 8e^{-3t} & 4e^{2t} - 2e^{2t} \end{pmatrix} = \begin{pmatrix} -3e^{-3t} & 2e^{2t} \\ 12e^{-3t} & 2e^{2t} \end{pmatrix}$$

✓

10. $\frac{d^2x_1}{dt^2} = -2x_1 + x_2$ $\qquad x_1' = x_3 \longrightarrow \frac{dx_3}{dt} = \frac{d^2x_1}{dt^2}$

$\frac{d^2x_2}{dt^2} = x_1 - 2x_2$ $\qquad x_2' = x_4 \longrightarrow \frac{dx_4}{dt} = \frac{d^2x_2}{dt^2}$

a. $\frac{dx_1}{dt} = x_3$

$\frac{dx_2}{dt} = x_4$

$\frac{dx_3}{dt} = -2x_1 + x_2$

$\frac{dx_4}{dt} = x_1 - 2x_2$

b. $\vec{X}' = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 1 & 0 & 0 \\ 1 & -2 & 0 & 0 \end{bmatrix} \vec{X}$

c. (use technology)

$\lambda_1 = \sqrt{3}\, i \quad \begin{bmatrix} \frac{1}{\sqrt{3}} i \\ -\frac{1}{\sqrt{3}} i \\ -1 \\ 1 \end{bmatrix}$ $\qquad \lambda_2 = -\sqrt{3}\, i \quad \begin{bmatrix} -\frac{1}{\sqrt{3}} i \\ \frac{1}{\sqrt{3}} i \\ -1 \\ 1 \end{bmatrix}$

$\lambda_3 = i \quad \begin{bmatrix} -i \\ -i \\ 1 \\ 1 \end{bmatrix}$ $\qquad \lambda_4 = -i \quad \begin{bmatrix} i \\ i \\ 1 \\ 1 \end{bmatrix}$

$\cos\sqrt{3}t, \sin\sqrt{3}t, \cos t, \sin t$ $\qquad$ modes $\Rightarrow \omega = \sqrt{3}, \omega = 1$

$$\begin{bmatrix} i \\ -i \\ -\sqrt{3} \\ \sqrt{3} \end{bmatrix} (\cos\sqrt{3}t + i\sin\sqrt{3}t) = \begin{bmatrix} i\cos\sqrt{3}t - \sin\sqrt{3}t \\ -i\cos\sqrt{3}t + \sin\sqrt{3}t \\ -\sqrt{3}\cos\sqrt{3}t - i\sqrt{3}\sin\sqrt{3}t \\ \sqrt{3}\cos\sqrt{3}t + i\sqrt{3}\sin\sqrt{3}t \end{bmatrix}$$

$$= c_1 \begin{bmatrix} -\sin\sqrt{3}t \\ \sin\sqrt{3}t \\ -\sqrt{3}\cos\sqrt{3}t \\ \sqrt{3}\cos\sqrt{3}t \end{bmatrix} + c_2 \begin{bmatrix} \cos\sqrt{3}t \\ -\cos\sqrt{3}t \\ -\sin\sqrt{3}t \cdot \sqrt{3} \\ \sqrt{3}\sin\sqrt{3}t \end{bmatrix}$$

$$\begin{bmatrix} -i \\ -i \\ 1 \\ 1 \end{bmatrix} (\cos t + i\sin t) = \begin{bmatrix} -i\cos t + \sin t \\ -\cos t + \sin t \\ \cos t + i\sin t \\ \cos t + i\sin t \end{bmatrix}$$

$$c_3 \begin{bmatrix} \sin t \\ \sin t \\ \cos t \\ \cos t \end{bmatrix} + c_4 \begin{bmatrix} -\cos t \\ -\cos t \\ \sin t \\ \sin t \end{bmatrix}$$

$$\vec{X} = c_1 \begin{bmatrix} -\sin\sqrt{3}t \\ \sin\sqrt{3}t \\ -\sqrt{3}\cos\sqrt{3}t \\ \sqrt{3}\cos\sqrt{3}t \end{bmatrix} + c_2 \begin{bmatrix} \cos\sqrt{3}t \\ -\cos\sqrt{3}t \\ -\sqrt{3}\sin\sqrt{3}t \\ \sqrt{3}\sin\sqrt{3}t \end{bmatrix} + c_3 \begin{bmatrix} \sin t \\ \sin t \\ \cos t \\ \cos t \end{bmatrix} + c_4 \begin{bmatrix} -\cos t \\ -\cos t \\ \sin t \\ \sin t \end{bmatrix}$$

e. $$\begin{bmatrix} -1 \\ 3 \\ 0 \\ 0 \end{bmatrix} = \cancel{c_1 \begin{bmatrix} 0 \\ 0 \\ -\sqrt{3} \\ \sqrt{3} \end{bmatrix}} + c_2 \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} + \cancel{c_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}} + c_4 \begin{bmatrix} -1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 1 & -1 & -1 \\ -1 & -1 & 3 \end{array}\right] \qquad c_2 = -2,\ c_4 = -1$$

$$\vec{X} = -2 \begin{bmatrix} \cos\sqrt{3}t \\ -\cos\sqrt{3}t \\ -\sqrt{3}\sin\sqrt{3}t \\ \sqrt{3}\sin\sqrt{3}t \end{bmatrix} - \begin{bmatrix} -\cos t \\ -\cos t \\ \sin t \\ \sin t \end{bmatrix}$$

yes, the solution is periodic since there is no damping (but since $\sqrt{3}$ & 1 are not integer multiples it won't look that way)

to graph plot the parametric equations
$X_1 = -2\cos\sqrt{3}t + \cos t$
$X_2 = 2\cos\sqrt{3}t + \cos t$

and separately
$X_3 = 2\sqrt{3}\sin\sqrt{3}t - \sin t$
$X_4 = -2\sqrt{3}\sin\sqrt{3}t - \sin t$

f. answers will vary.