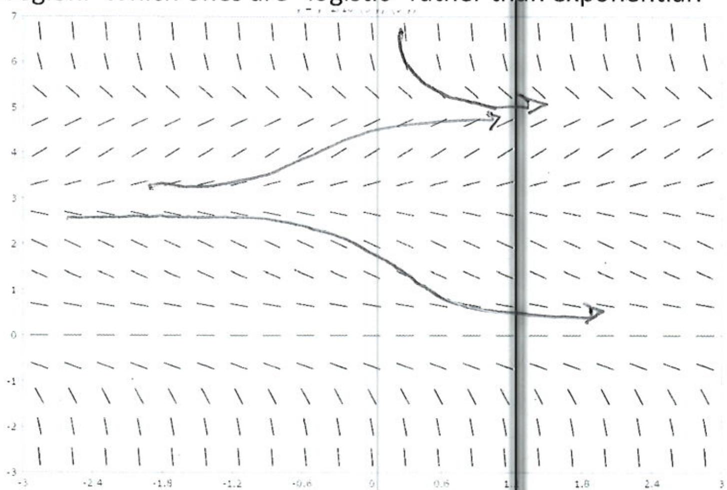
MTH 291, Skill Problems #5, Fall 2017 Name KEY

Instructions: Work problems on a separate sheet of paper and attach work to this page. You should show all work to receive full credit for problems. Checking your work with computer algebra systems is fine, but that doesn't count as "work" since you won't be able to use CAS programs on exams or quizzes. Graphs and longer answers that won't fit here, indicate which page of the work the answer can be found on and be sure to clearly indicate it on the attached pages.

1. Newton's law of cooling states that the temperature of an object changes at a rate proportional to the difference between its temperature and that of its surroundings. Suppose that the temperature of a cup of coffee obeys Newton's law of cooling. If the coffee has a temperature of 200° when freshly poured, and 1 minute later cooled to 190° in a room at 70°, determine when the coffee reaches a temperature of 150° (in minutes).

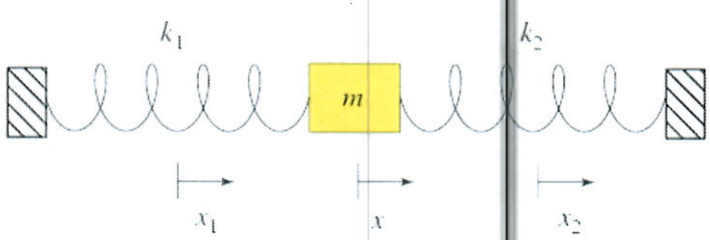
2. A direction field for the differential equation $y' = -\frac{1}{10}y^2(3-y)(5-y)$ is shown below. For each stationary point, describe it as a carrying capacity, a threshold, or neither. Graph a sample trajectory in each region. Which ones are "logistic" rather than exponential?



3. A spring with spring constant 4N/m is attached to a 1kg mass with negligible friction. If the mass is initially displaced to the right of equilibrium by 0.5m and has an initial velocity of 1 m/s toward equilibrium. Compute the amplitude of the oscillation.

4. A 16 pound weight is attached to a spring with friction constant 8lb · s/ft and spring constant 7lb/ft. Write the associated spring/mass ODE.

5. Using the image below, write a differential equation to model the location of the mass over time assuming no damping.



MTH 291 Skills #5 Key

1. $T(0) = 200$ $\quad C = 70°$
$T(1) = 190$

$$\frac{dT}{dt} = k(T-C)$$

$$\int \frac{dT}{T-C} = \int k\,dt \Rightarrow \ln|T-C| = kt + C \Rightarrow T - C = Ae^{kt} \Rightarrow T(t) = Ae^{kT} + C$$

$T(0) = 200 = A + C$
$T(1) = 190 = Ae^k + C$

$$\begin{array}{r} 200 = A + C \\ -190 = -Ae^k + C \\ \hline 10 = A(1-e^k) \end{array}$$

$A = 200 - 70 = 130$

$10 = 130(1-e^k)$

$\frac{1}{13} = 1 - e^k$

$+\frac{12}{13} = e^k$

$k = -.08$

$T(t) = 70 + 130e^{-.08t}$

$T(t) = 150$

$$\begin{array}{r} 150 = 70 + 130e^{-.08t} \\ -70 \quad -70 \\ \hline 80 = 130e^{-.08t} \end{array}$$

$\frac{80}{130} = \frac{8}{13} = e^{-.08t} \Rightarrow \ln\left(\frac{8}{13}\right) = -.08t \Rightarrow t = 6.07$ minutes

2. $y = 0$ semistable (neither)
$y = 3$ unstable - threshold
$y = 5$ stable - carrying capacity
logistic between 3 & 5, & 0 and 3

3. $\frac{dx}{dt} = y$ $\quad m = 1$ $\quad x(0) = .05$
$\quad k = 4$ $\quad y(0) = 1$

$\frac{dy}{dt} = -\frac{k}{m}x - \cancel{\frac{\gamma}{m}y}$ $\quad \gamma \approx 0 \Rightarrow \frac{d^2x}{dt^2} = -\frac{k}{m}x$

3 cont'd.

$X = A\sin(\omega t) + B\cos(\omega t)$

$X' = A\omega\cos(\omega t) - B\omega\sin(\omega t)$

$X'' = -A\omega^2\sin(\omega t) - B\omega\cos(\omega t)$

$X(0) = .5 \Rightarrow A\cancel{\sin(\omega(0))}^{\to 0} + B\cos(\omega(0)) = 0.5$

$.5 = B$

$X'(0) = A\omega\cos(\omega(0)) - B\omega\cancel{\sin(\omega(0))}^{\to 0} = 1$

$A\omega = 1 \qquad \Rightarrow \omega = \frac{k}{m} = 4$

$4A = 1 \Rightarrow A = 1/4$

$X(t) = 1/4\sin(4t) + \frac{1}{2}\cos(4t)$

4. $\frac{dx}{dt} = Y$ $\qquad k = 7$ $\qquad m = 1/2 \quad \approx 16 = m(32)$

$\gamma = 8 \qquad \Rightarrow m = 1/2$

$\frac{dy}{dt} = -\frac{k}{m}X - \frac{\gamma}{m}Y \longrightarrow$

$\frac{dx}{dt} = Y$

$\frac{dy}{dt} = -14X - 16Y$

5. $\frac{dx}{dt} = Y$

$\frac{dy}{dt} = -\frac{(k_1+k_2)}{m}X$

6. a. $u'' + \frac{1}{2}u' + 2u = 0$ $\qquad X_1 = u \quad X_2 = u' \to X_2' = u''$

$X_1' = X_2$

$X_2' = -2X_1 - \frac{1}{2}X_2$

b. $u^{iv} - u = 0$ $\qquad X_1 = u, \; X_2 = X_1' = u' \quad X_3 = X_2' = u'' \quad X_4 = X_3' = u'''$

6b cont'd.

$$x_1' = x_2$$
$$x_2' = x_3$$
$$x_3' = x_4$$
$$x_4' = x_1$$

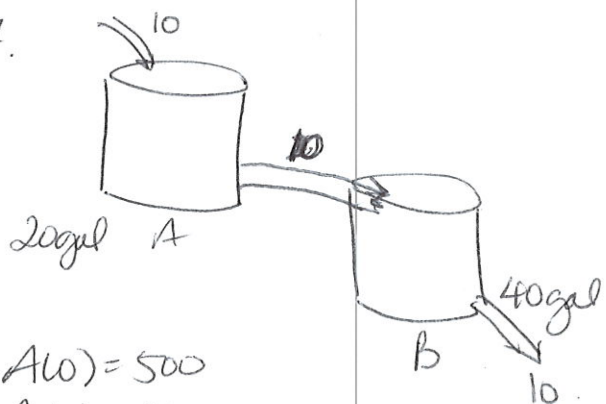
c. $t^2u'' + tu' + (t^2 - \frac{1}{4})u = 0$ $\quad x_1 = u,\ x_1' = x_2 = u'\ \ x_2' = u''$

$$x_1' = x_2 \qquad\qquad x_1' = x_2$$
$$t^2 x_2' = -tx_2 - (t^2 - \tfrac{1}{4})x_1 \rightarrow x_2' = \left(\frac{\frac{1}{4} - t^2}{t^2}\right)x_1 - \frac{1}{t}x_2$$

d. $u'' + \frac{1}{4}u' + 4u = 2\cos 3t,\ u(0) = 1,\ u'(0) = -2$

$$x_1' = x_2 \qquad x_1 = u,\ x_1' = x_2 = u'\ \ x_2' = u''$$
$$x_2' = -4x_1 - \tfrac{1}{4}x_2 + 2\cos 3t \qquad x_1(0) = 1,\ x_2(0) = -2$$

7.

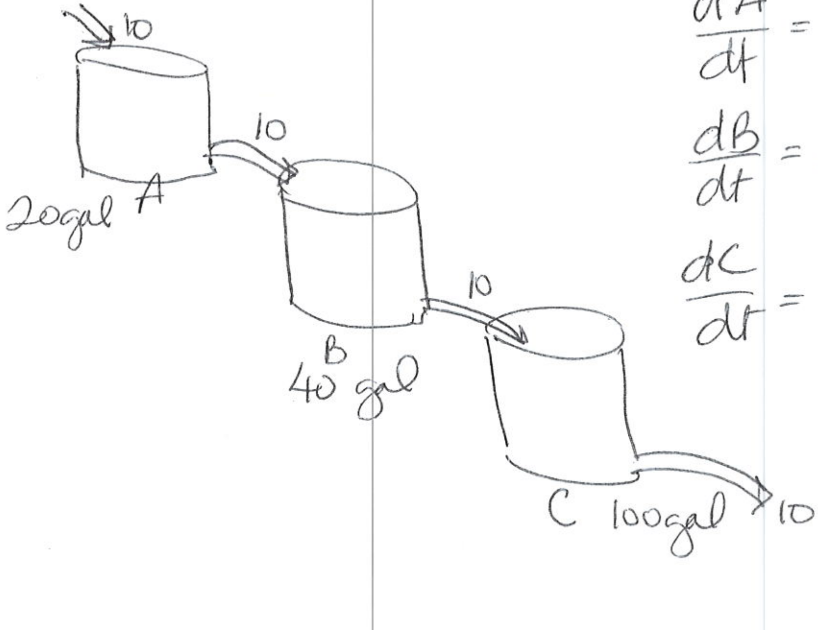


$A(0) = 500$
$B(0) = 500$

$$\frac{dA}{dt} = -\frac{A}{20}\cdot\frac{10\text{gal}}{\text{min}} \Rightarrow \frac{dA}{dt} = -\frac{1}{2}A$$

$$\frac{dB}{dt} = \frac{1}{2}A - \frac{B}{40}\cdot\frac{10\text{gal}}{\text{min}} \Rightarrow \frac{dB}{dt} = \frac{1}{2}A - \frac{1}{4}B$$

8.



$$\frac{dA}{dt} = -\frac{1}{2}A$$
$$\frac{dB}{dt} = \frac{1}{2}A - \frac{1}{4}B$$
$$\frac{dC}{dt} = \frac{1}{4}B - \frac{1}{10}C$$