MTH 291, Exam #2, Part I, Fall 2017 Name KEY

Instructions: Show all work. You will earn full credit for correct answers only when accompanied by work or explanation. Answers that are incorrect and have no work will not receive any partial credit. Use exact answers, except in applied problems: round to two decimal places, or the number requested in the problem.

1. Find the solutions to the systems.

a. $\vec{x}' = \begin{bmatrix} -4 & 2 \\ -\frac{5}{2} & 2 \end{bmatrix}\vec{x}$

$(-4-\lambda)(2-\lambda)+5=0$

$\lambda^2+2\lambda-8+5=0$

$\lambda^2+2\lambda-3=0$

$(\lambda+3)(\lambda-1)=0$

$\lambda=-3,\ \lambda=1$

$\lambda_1=-3$

$\begin{bmatrix} -1 & 2 \\ -5/2 & 5 \end{bmatrix}$ $-x_1+2x_2=0$, $x_1=2x_2$, $\vec{v}_1=\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\lambda_2=1$

$\begin{bmatrix} -5 & 2 \\ -5/2 & 1 \end{bmatrix}$ $-5x_1+2x_2=0$, $x_1=\frac{2}{5}x_2$, $\vec{v}_2=\begin{bmatrix} 2 \\ 5 \end{bmatrix}$

$$\vec{x}=c_1\begin{bmatrix} 2 \\ 1 \end{bmatrix}e^{-3t}+c_2\begin{bmatrix} 2 \\ 5 \end{bmatrix}e^{t}$$

b. $\vec{x}' = \begin{bmatrix} 5 & 1 \\ -2 & 3 \end{bmatrix}\vec{x}$

$(5-\lambda)(3-\lambda)+2=0$

$\lambda^2-8\lambda+15+2=0$

$\lambda^2-8\lambda+17=0$

$\lambda=\dfrac{8\pm\sqrt{64-68}}{2}$

$=\dfrac{8\pm 2i}{2}=4\pm i$

$\begin{bmatrix} 5-(4+i) & 1 \\ -2 & 3-(4+i) \end{bmatrix}=\begin{bmatrix} 1-i & 1 \\ -2 & -1-i \end{bmatrix}$

$-2x_1-(1+i)x_2=0$

$x_1=\dfrac{1+i}{-2}x_2$ $\vec{v}_1=\begin{bmatrix} 1+i \\ -2 \end{bmatrix}$

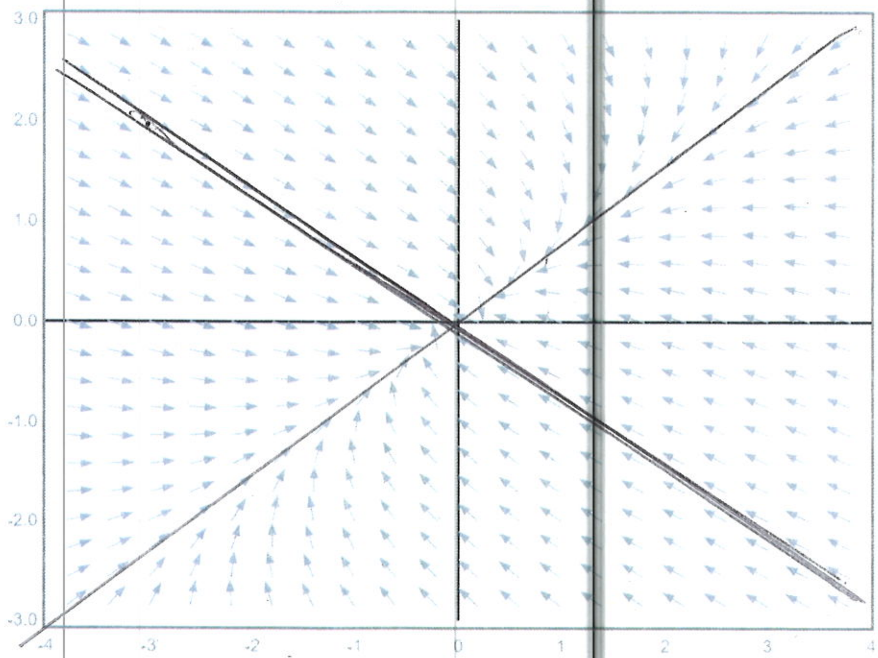
$e^{4t}\begin{bmatrix} 1+i \\ -2 \end{bmatrix}(\cos t+i\sin t)=$

$e^{4t}\begin{bmatrix} 1\cos t+i\sin t+i\cos t-\sin t \\ -2\cos t-2i\sin t \end{bmatrix}$

$$\vec{x}=c_1e^{4t}\begin{bmatrix} \cos t-\sin t \\ -2\cos t \end{bmatrix}+c_2e^{4t}\begin{bmatrix} \sin t+\cos t \\ -2\sin t \end{bmatrix}$$

2. A phase plane for a linear system of equations is shown below. Use the graph to estimate the two straight-line solutions for the system. [Hint: a straight edge may help.] Characterize the origin as a repeller, attractor or saddle point.

$y = -\frac{2}{3}x$

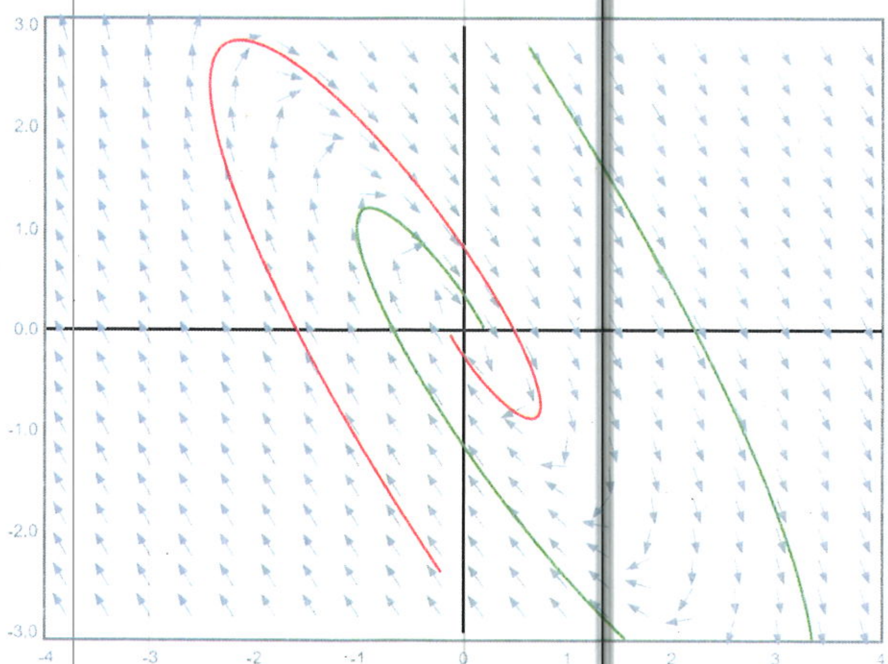


$\frac{3}{4}$

$y = \frac{3}{4}x$

attractor

3. The phase plane of a system of linear differential equations is shown below along with a couple sample trajectories. If the system is modeling a spring, describe the spring system in terms of its damping.



underdamped

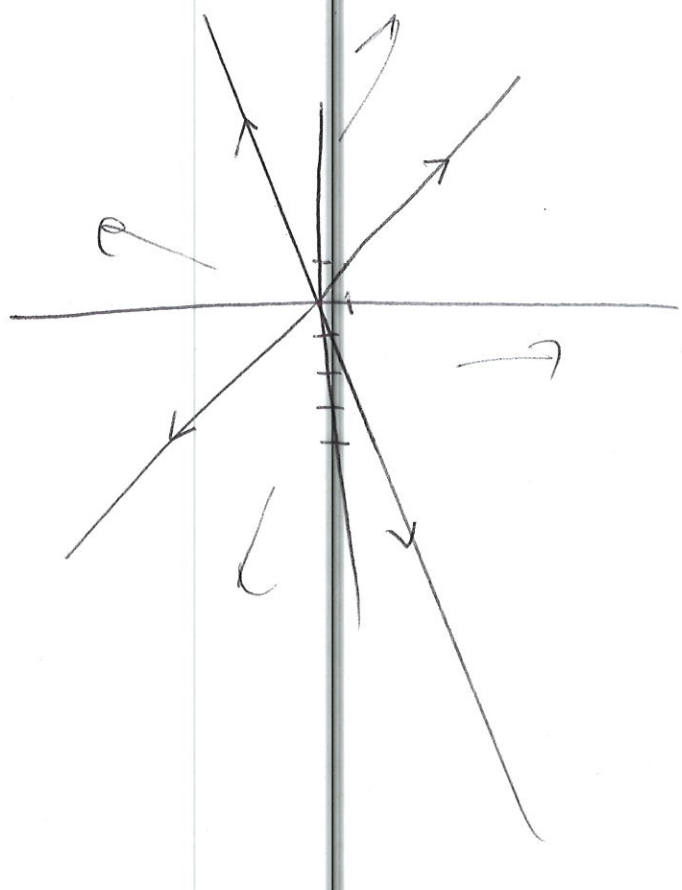
4. The system $\vec{x}' = \begin{bmatrix} 6 & 1 \\ 4 & 3 \end{bmatrix} \vec{x}$ has the solution $\vec{x} = c_1 \begin{bmatrix} 1 \\ -4 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{7t}$. Consider the initial conditions $\begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} -8 \\ 3 \end{bmatrix}$. Find the linear combination of the straight-line solutions that satisfy the initial conditions. Sketch the phase plane.

$$c_1 + c_2 = -8$$
$$-4c_1 + c_2 = 3$$

$$\left[\begin{array}{cc|c} 1 & 0 & -11/5 \\ 0 & 1 & -29/5 \end{array}\right]$$

$$c_1 = -11/5$$
$$c_2 = -29/5$$

$$\vec{x} = \begin{bmatrix} -11/5 \\ 44/5 \end{bmatrix} e^{2t} + \begin{bmatrix} -29/5 \\ -29/5 \end{bmatrix} e^{7t}$$



MTH 291, Exam #2, Part II, Fall 2017 Name KEY

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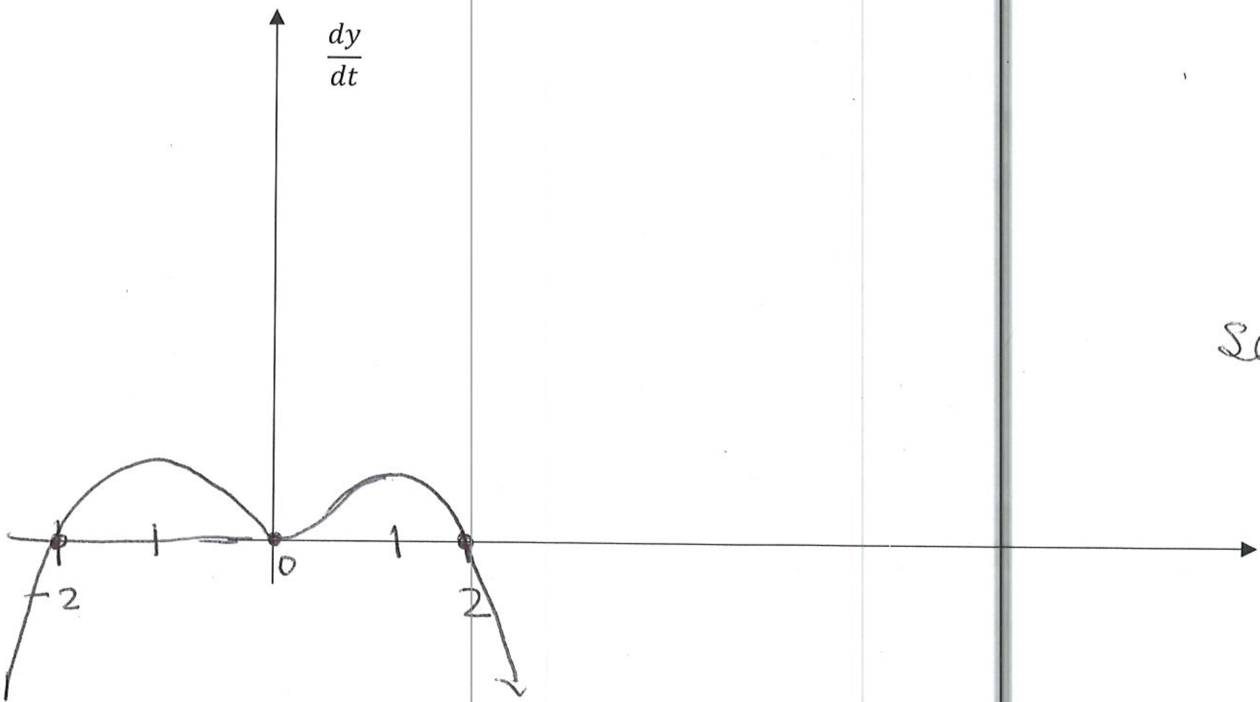
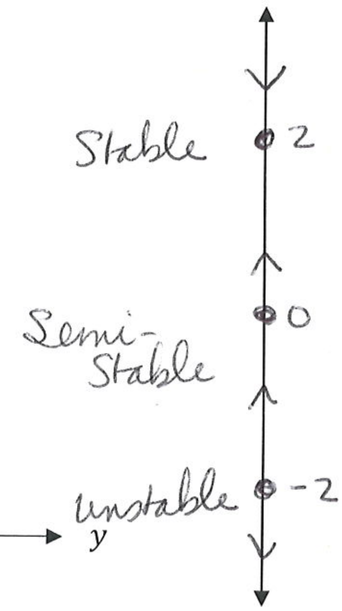
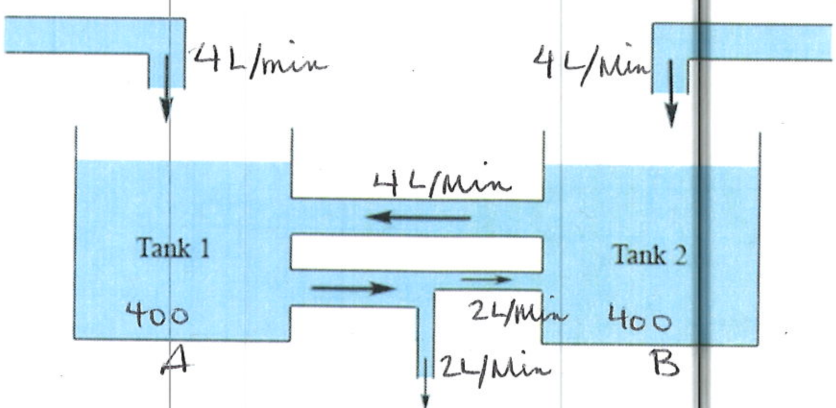
1. Sketch the phase plane of $\frac{dy}{dt} = y^2(4 - y^2)$ on the axis below, and then convert that to a phase line.





2. Set up a system of linear differential equations to model the coupled tank problem below. The flow of pure water into Tank 1 and Tank 2 is 4 L/min. Tank 1 initially contains 30 kg of salt in solution, and Tank 2 initially contains 10 kg of salt. Both tanks contain 400 L at $t = 0$. The flow from Tank 2 to Tank 1 is 4 L/min. The flow from Tank 1 to Tank 2 is 2 L/min, and the flow from Tank 1 into the environment is 2 L/min. Write the system with initial conditions. Do not solve.



$A(0) = 30$ kg

$B(0) = 10$ kg

$$\frac{dA}{dt} = 0 - \frac{A\cdot 4}{400+2t} + \frac{B\cdot 4}{400+2t}$$

$$\frac{dB}{dt} = 0 - \frac{B\cdot 4}{400+2t} + \frac{A\cdot 2}{400+4t}$$

3. Consider the system of differential equations:

$$\begin{cases} \dfrac{dx}{dt} = 4x - 7y \\ \dfrac{dy}{dt} = 5x \end{cases}$$

Sketch the phase plane by first drawing the nullclines, and then filling in key vectors in each region of the vector field. Is the equilibrium stable, unstable or a saddle point?

unstable

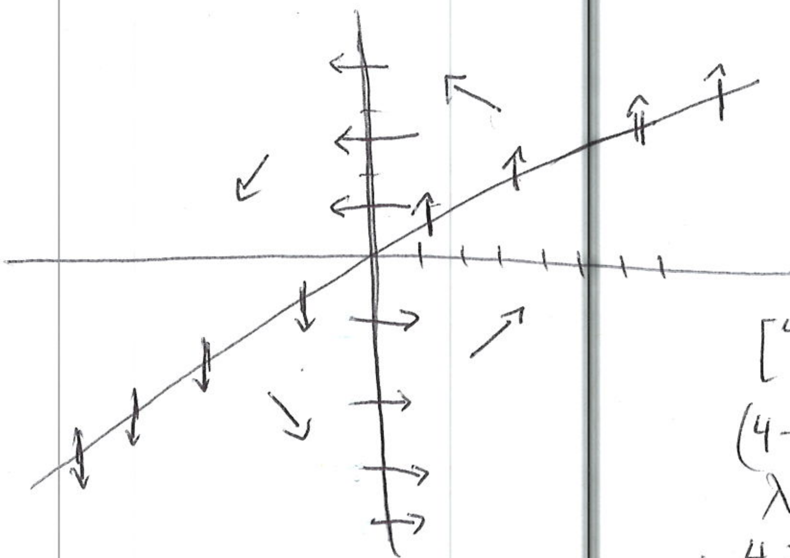
$0 = 4x - 7y$

$7y = 4x$

$y = \frac{4}{7}x$

$0 = 5x$

$x = 0$



$$\begin{bmatrix} 4 & -7 \\ 5 & 0 \end{bmatrix}$$

$(4-\lambda)(-\lambda)+35 = 0$

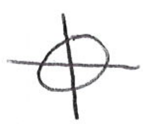
$\lambda^2 - 4\lambda + 35 = 0$

$\lambda = \dfrac{4 \pm \sqrt{16 - 140}}{2} = 2 \pm \sqrt{31}\, i$

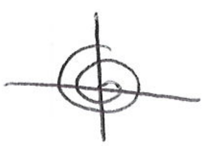
real positive

4. Draw examples of solutions in the phase plane $\left(x \; vs. \frac{dx}{dt}\right)$ for a spring system with the following properties:

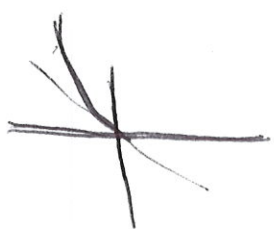
a. Undamped



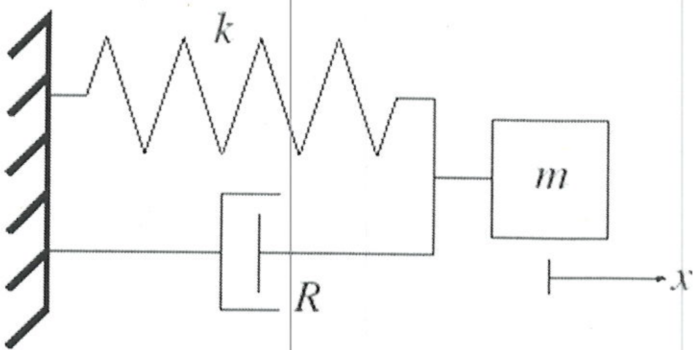
b. Underdamped



c. Overdamped



5. Set up the system of differential equations for the spring-mass system below if $k = 3, m = 12, R = b = \gamma = 2$. Use technology to graph a phase plane and determine if the solution is underdamped or overdamped.



$$\frac{dx}{dt} = y$$

$$\frac{dy}{dt} = -\frac{3}{12}x - \frac{2}{12}y$$

$$\Rightarrow \frac{dx}{dt} = y$$

$$\frac{dy}{dt} = -\frac{1}{4}x - \frac{1}{6}y$$

or $\vec{x}' = \begin{bmatrix} 0 & 1 \\ -1/4 & -1/6 \end{bmatrix} \vec{x}$

6. Use Euler's method for systems to estimate the solution for the system $\begin{cases} x' = 2x - y & x(0) = 6 \\ y' = x & 'y(0) = 2' \end{cases}$ beginning at $t = 0$, with $\Delta t = 0.2$ to estimate $\begin{bmatrix} x(1) \\ y(1) \end{bmatrix}$.

$f_{x_0} = 2(6) - 2 = 10$

$f_{y_0} = 6$

$X_{n+1} = X_1 = 10(.2) + 6 = 8$

$Y_{n+1} = Y_1 = 6(.2) + 2 = 3.2$

continued in Excel file attached

$X_5 \approx X(1) = 126.64$

$Y_5 \approx y(1) = 87.94$

7. A thermometer reading $70°\ F$ is placed in an oven preheated to a constant temperature. Through a glass window in an oven door, an observer records that the thermometer reads $110°\ F$ after ½ minute and $145°\ F$ after 1 minute. How hot is the oven?

$T(0) = 70$

$$\frac{dT}{dt} = k(T - T_0)$$

oven is 390°

$$\int \frac{dT}{T - T_0} = \int k\, dt$$

$$\ln|T - T_0| = kt + C$$

$$T - T_0 = Ae^{kt}$$

$$T = Ae^{kt} + T_0$$

$$70 = Ae^0 + T_0 \longrightarrow 70 = A + T_0 \rightarrow 70 = -320 + T_0$$

$$\boxed{390 = T_0}$$

$$110 = Ae^{k(1/2)} + T_0$$

$$145 = Ae^{k} + T_0$$

$$40 = A(e^{1/2k} - 1)$$

$$35 = A(e^k - e^{1/2k})$$

$$A = \frac{35}{e^k - e^{1/2k}} \qquad A = \frac{40}{e^{1/2k} - 1} \Rightarrow A = \frac{40}{7/8 - 1} = -320$$

$$\frac{35}{e^k - e^{1/2k}} = \frac{40}{e^{1/2k} - 1} \Longrightarrow 35e^{1/2k} - 35 = 40e^k - 40e^{1/2k}$$

$$0 = 40e^k - 75e^{1/2k} + 35$$

$$0 = 8e^k - 15e^{1/2k} + 7$$

$$(8e^{1/2k} - 7)(e^{1/2k} - 1)$$

$$e^{1/2k} = 7/8 \qquad e^{1/2k} = 1 \Rightarrow k = 0$$

X

$$k = 2\ln(7/8)$$