MTH 291, Exam #1, Fall 2017 Name KEY

Instructions: Show all work. You will earn full credit for correct answers only when accompanied by work or explanation. Answers that are incorrect and have no work will not receive any partial credit. Use exact answers, except in applied problems: round to two decimal places, or the number requested in the problem.

1. Verify that $y = (1 - \sin x)^{-1/2}$ is a solution to the differential equation $2y' = y^3 \cos x$. (10 points)

$$y' = -\tfrac{1}{2}(1-\sin x)^{-3/2}(-\cos x) = \tfrac{1}{2}(1-\sin x)^{-3/2}\cos x$$

$$2y' = (1-\sin x)^{-3/2}\cos x \longleftrightarrow$$

$$y^3\cos x = \left[(1-\sin x)^{-1/2}\right]^3\cos x = (1-\sin x)^{-3/2}\cos x$$

they are equal, so it is a solution.

2. Verify that the pair of function $x = e^{-2t} + 3e^{6t}$ and $y = -e^{-2t} + 5e^{6t}$ is a set of solutions to the system $\frac{dx}{dt} = x + 3y$ and $\frac{dy}{dt} = 5x + 3y$. (12 points)

$$\frac{dx}{dt} = -2e^{-2t} + 18e^{6t} \qquad x+3y = e^{-2t} + 3e^{6t} + 3(-e^{-2t}+5e^{6t})$$

$$\longleftrightarrow \qquad = e^{-2t} + 3e^{6t} - 3e^{-2t} + 15e^{6t}$$

$$= -2e^{-2t} + 18e^{6t}$$

$$\frac{dy}{dt} = 2e^{-2t} + 30e^{6t} \qquad 5x+3y = 5(e^{-2t}+3e^{6t}) + 3(-e^{-2t}+5e^{6t})$$

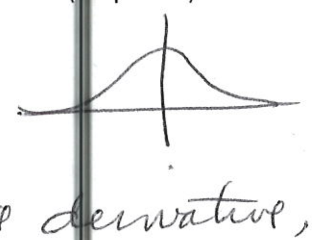
$$\longleftrightarrow \qquad = 5e^{-2t} + 15e^{6t} - 3e^{-2t} + 15e^{6t}$$

it is a solution

$$= 2e^{-2t} + 30e^{6t}$$

3. Consider the differential equation $\frac{dy}{dx} = e^{-x^2}$. Explain why a solution of the differential equation must be an increasing function on any interval of the x-axis. (10 points)

e^{-x^2} is always positive



So any solution to this ODE must have a positive derivative, i.e. be increasing everywhere

4. Write a differential equation to model a situation where a population growth is proportional to the square of the population. (8 points)

$$\frac{dP}{dt} = kP^2$$

5. Suppose that a large mixing tank initially holds 300 gallons of water, into which is dissolved 50 pounds of salt. Another brine solution is pumped into the tank at a rate of 3 gal/min, and when the solution is well-stirred, it is then pumped out at a slower rate of 2 gal/min.
   a. If the concentration of the solution entering is 2 lbs/gal, determine a differential equation for the amount of salt at time t. (10 points)

$$\boxed{S(0) = 50}$$

$$\frac{dS}{dt} = \frac{3\,\text{gal}}{\text{min}} \cdot \frac{2\,\text{lbs}}{\text{gal}} - \frac{2\,\text{gal}}{\text{min}}\,\frac{S}{300+t\ \text{gal}}$$

$$\boxed{\frac{dS}{dt} = 6 - \frac{2S}{300+t}}$$

$t = 700$ minutes

   b. Solve the equation you found. How long is the equation defined? Assume the tank holds 1000 gallons of water. (8 points)

$$\frac{dS}{dt} + \frac{2}{300+t} S = 6 \qquad \mu = e^{\int \frac{2}{300+t}\,dt} = e^{2\ln(300+t)} = e^{\ln(300+t)^2} = (300+t)^2$$

$$(300+t)^2 \frac{dS}{dt} + 2(300+t)S = 6(300+t)^2 \qquad \boxed{S = 2(300+t) + \frac{C}{(300+t)^2}}$$

$$\int [(300+t)^2 S]' = \int 6(300+t)^2\,dt = 2(300+t)^3 + C$$

   c. If the solution is defined for all positive time, find the limit of the concentration. If the equation predicts the tank overflows, find the concentration at that time. (8 points)

$$50 = 600 + 2(0) + \frac{C}{(300+0)^2}$$

$$50 = 600 + \frac{C}{90{,}000}$$

$$-550 = \frac{C}{90{,}000}$$

$$C = -4.95\times10^7$$

$$S = 600 + 2t - \frac{4.95\times10^7}{(300+t)^2}$$

tank overflows at 700 minutes

$$S = 2(300+700) - \frac{4.95\times10^7}{(300+700)^2}$$

$$S = 1950.5$$

$$\frac{S}{\text{gal}} = \frac{1950.5}{1000} = 1.95\ \text{g/gal}$$

6. Identify the slope fields that match each differential equation. (4 points each)

i. $\frac{dy}{dx} = x^2 - y^2$

ii. $\frac{dy}{dx} = (\sin x)(\cos x)$

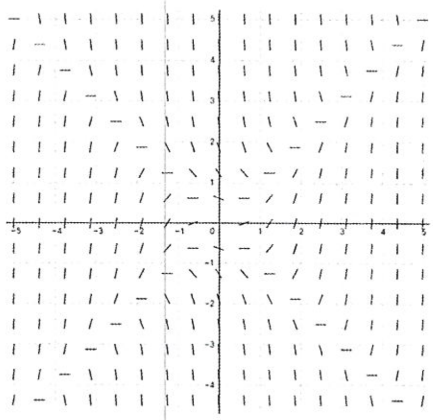
iii. $\frac{dy}{dx} = e^{-0.1xy^2}$

iv. $\frac{dy}{dx} = \frac{1}{y} - x$

v. $\frac{dy}{dx} = 1 - xy$

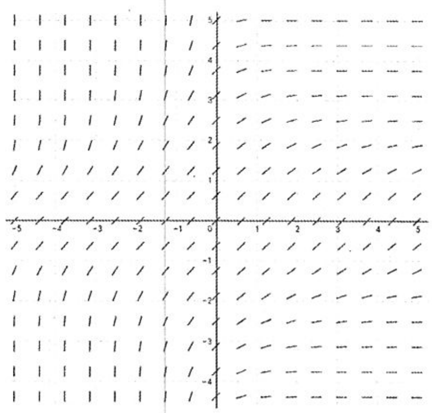
vi. $\frac{dy}{dx} = x^{2/3} - x \sin y$

i



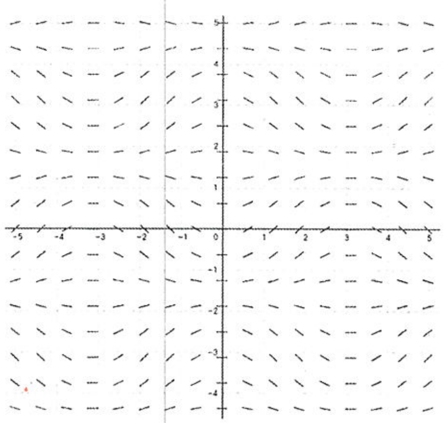
a.

iii

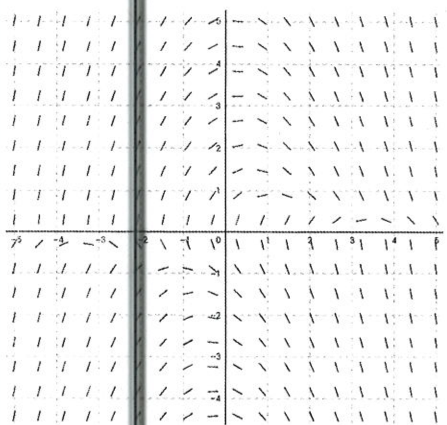


b.

ii

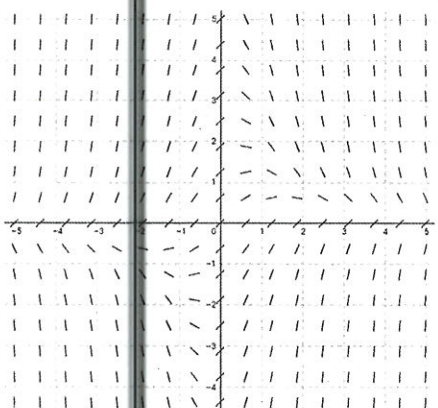


c.



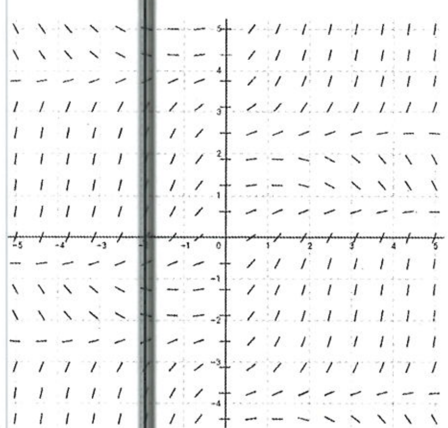
d.

IV



e.

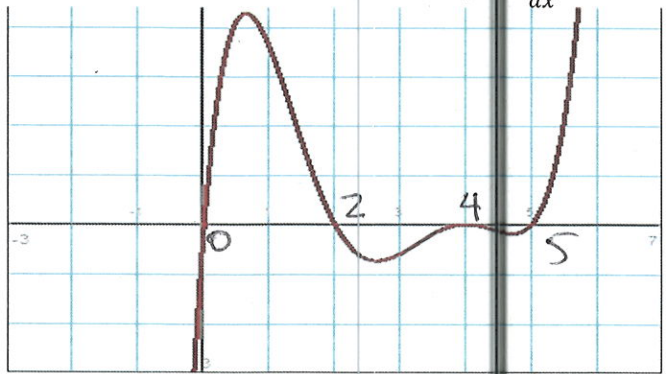
V

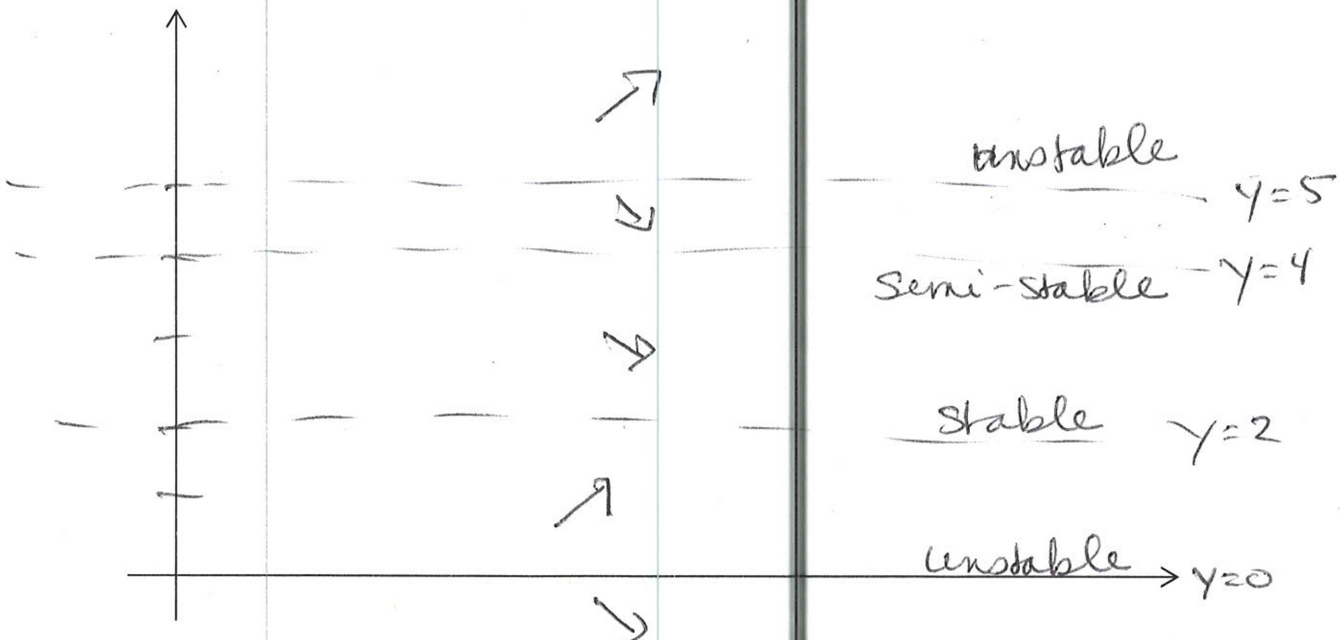


f.

Vi

7. a. Sketch the slope field consistent with the phase plane of y vs. $\frac{dy}{dx}$ shown below. (8 points)





b. What would the phase line look like? (7 points)

8. Solve the differential equation $e^x y\frac{dy}{dx} = e^{-y} + e^{-2x-y}$ by separation of variables. (12 points)

$$e^x y \frac{dy}{dx} = e^{-y}(1+e^{-2x})$$

$$ye^y\,dy = e^{-x}(1+e^{-2x})\,dy$$

$$\int ye^y\,dy = \int e^{-x} + e^{-3x}\,dx$$

$u = y \quad dv = e^y$

$du = dy \quad v = e^y$

$ye^y - \int e^y dy$

$$ye^y - e^y = -e^{-x} - \frac{1}{3}e^{-3x} + C$$

$$(y-1)e^y = -e^{-x} - \frac{1}{3}e^{-3x} + C$$

9. Solve the linear differential equation $x\frac{dy}{dx} - y = x^2 \sin x$ using the method of integrating factors. (12 points)

$$\mu = e^{\int -\frac{1}{x}dx} = e^{-\ln x} = \cancel{e}^{\ln \frac{1}{x}} = \frac{1}{x}$$

$$\frac{dy}{dx} - \frac{1}{x}y = x\sin x$$

$$\frac{1}{x}\cdot\frac{dy}{dx} - \frac{1}{x^2}y = \sin x$$

$$\int \left(\frac{1}{x}y\right)' = \int \sin x\,dx$$

$$\frac{1}{x}y = -\cos x + C$$

$$y = -x\cos x + Cx$$

10. Verify that the differential equation $(x^2 + 2y^2)dx = xydy$ is homogeneous. Use an appropriate substitution to make it separable. (8 points)

$x \to tx;\ y \to ty$

$t^2x^2 + 2t^2y^2 = t^2(x^2+2y^2)$ match $\Rightarrow$ homogeneous

$txty = t^2(xy)$

$\frac{dy}{dx} = \frac{x^2+2y^2}{xy}$ $\quad y = vx$ $\quad y' = v'x + v$

$\boxed{\frac{v}{1+v^2} \cdot dv = \frac{1}{x}dx}$

$v'x + v = \frac{x^2+2v^2x^2}{x^2 v} = \frac{\cancel{x^2}(1+2v^2)}{\cancel{x^2}v}$

$v'x = \frac{1+2v^2}{v} - v\frac{v}{v} = \frac{1+2v^2-v^2}{v} = \frac{1+v^2}{v}$

11. Rewrite the Bernoulli differential equation $3(1+t^2)\frac{dy}{dt} = 2ty(y^3 - 1)$ with an appropriate substitution to make the equation linear. [Hint: distribution and put in standard form first.] (8 points)

$3(1+t^2)\frac{dy}{dt} = 2ty^4 - 2ty$

$\frac{dy}{dt} + \frac{2t}{3(1+t^2)}y = \frac{2t}{3(1+t^2)}y^4$

$-3y^{-4}\frac{dy}{dt} - \frac{\cancel{6}^{2}t}{\cancel{3}(1+t^2)}y^{-3} = \frac{-\cancel{6}^{2}t}{\cancel{3}(1+t^2)}$

$y^n = y^4 \to n = 4$

$(1-n)y^{-n} = (-3)y^{-4}$

$y^{-3} = z$

$-3y^{-4}\frac{dy}{dt} = z'$

$$\boxed{z' - \frac{2t}{1+t^2}z = \frac{-2t}{1+t^2}}$$

12. Use Euler's method to estimate $y(1.5)$ for the differential equation $y' = xy^2 - \frac{y}{x}$ if $y(1) = 1$. Estimate the value using $\Delta t = .05$. Calculate one step by hand, but you can do the remaining points in Excel. (14 points)

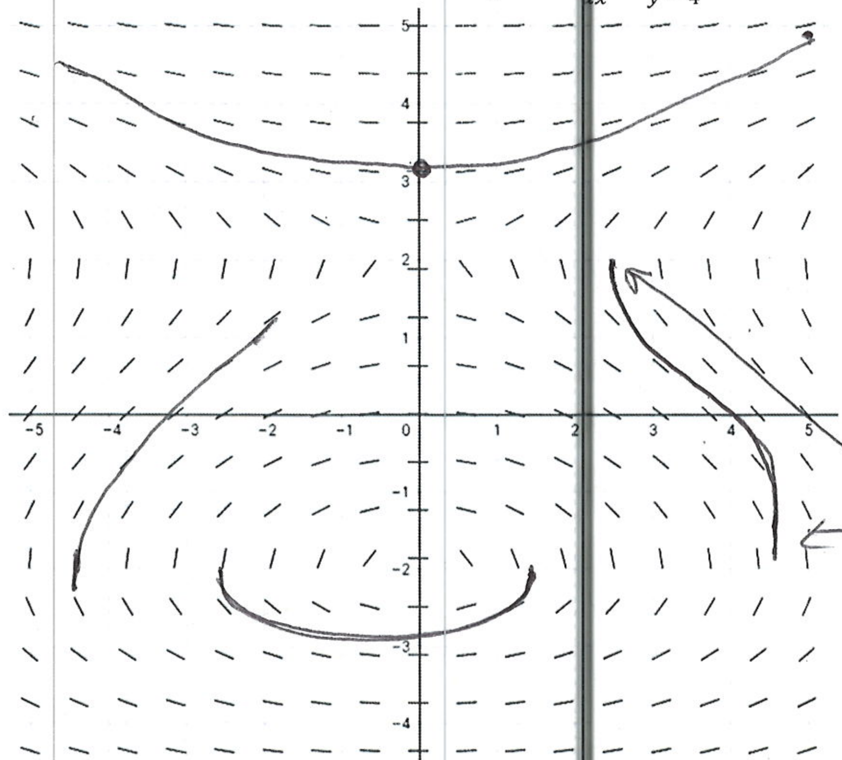
n	t_n	y_n	$m_n = y_n'$	y_{n+1}
0	1	1	$(1)(1)^2 - \frac{1}{1} = 0$	$0(.05) + 1 = 1$
1	1.05	1	$(1.05)(1)^2 - \frac{1}{1.05} = .0976$	$(.0976)(.05)+1 = 1.00488$
2	1.10	1.00488		

$\vdots$

$y(1.5) \approx 1.26957$

Step (n)	t_n	y_n	m_n=f(t_n,y_n)	Delta_t=h	y_(n+1)
0	1	1	0	0.05	1
1	1.05	1	0.097619048	0.05	1.004881
2	1.1	1.004881	0.197236163	0.05	1.014743
3	1.15	1.014743	0.301773291	0.05	1.029831
4	1.2	1.029831	0.414470463	0.05	1.050555
5	1.25	1.050555	0.539138165	0.05	1.077512
6	1.3	1.077512	0.680486067	0.05	1.111536
7	1.35	1.111536	0.844581938	0.05	1.153765
8	1.4	1.153765	1.039525934	0.05	1.205742
9	1.45	1.205742	1.276482507	0.05	1.269566
10	1.5	1.269566	1.5713184	0.05	1.348132

13. The slope field shown below is from the differential equation $\frac{dy}{dx} = \frac{x}{y^2-4}$.



vertical tangent!

answers will vary

Choose three initial conditions and draw approximate (qualitative) solution curves for those conditions (backwards and forwards) in time. Be alert for vertical asymptotes that might violate the uniqueness conditions. Each curve should have different behavior. (9 points)

14. Apply the existence and uniqueness theorem to the differential equations $\frac{dy}{dt} = e^{y-t}(\sec y)(1+t^2)$. Determine where a unique solution is guaranteed to exist. Sketch the region in the plane. (10 points)

$\frac{dy}{dt} = f(t,y)$

$e^{y-t}(\sec y)(1+t^2)$ defined on all reals except odd multiples of $\pi/2$ (in y) (for all t)

$\frac{\partial f}{\partial y} = e^{y-t}(\sec y)(1+t^2)$
$+ e^{y-t}(\tan y \sec y)(1+t^2)$. ← no new issues

15. Consider the competition model below.

$$\frac{dx}{dt} = x(2 - 0.4x - 0.3y)$$
$$\frac{dy}{dt} = y(1 - 0.1y - 0.3x)$$

How is the system changing at each of the following pairs of initial conditions?

a. $x(0) = 1.5, y(0) = 3.5$ (5 points)

$\frac{dx}{dt} = .525$ $\quad \frac{dy}{dt} = .7$ both pops increasing

b. $x(0) = 4.5, y(0) = 0.5$ (5 points)

$\frac{dx}{dt} = .225$ $\quad \frac{dy}{dt} = -.2$ x increases while y decreases

c. What is the overall behavior of the system? Write a sentence or two to describe it. Feel free to use technology, plot additional points, find how the system behaves when a species disappears, or any place where a derivative is zero, to inform your answer. (12 points)

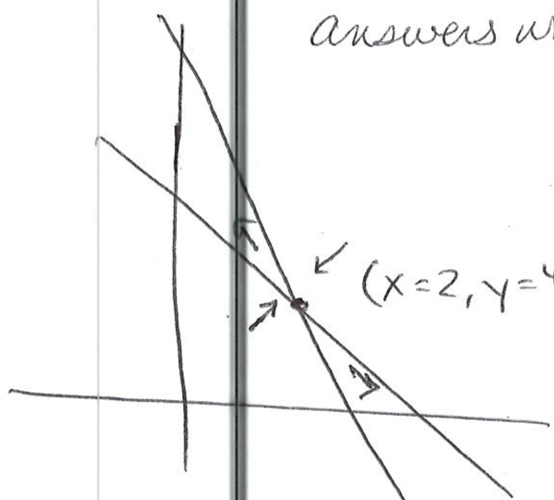
answers will vary

$2 - .4x - .3y = 0$

$\frac{2 - .4y}{.3} = y$

$1 - .1y - .3x = 0$

$\frac{1 - .3x}{.1} = y$



(x=2, y=4)

Populations reach an equilibrium when x=2, y=4

point is a saddle point