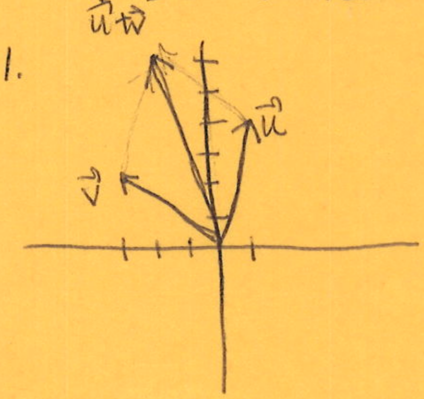
202 Homework #4 key

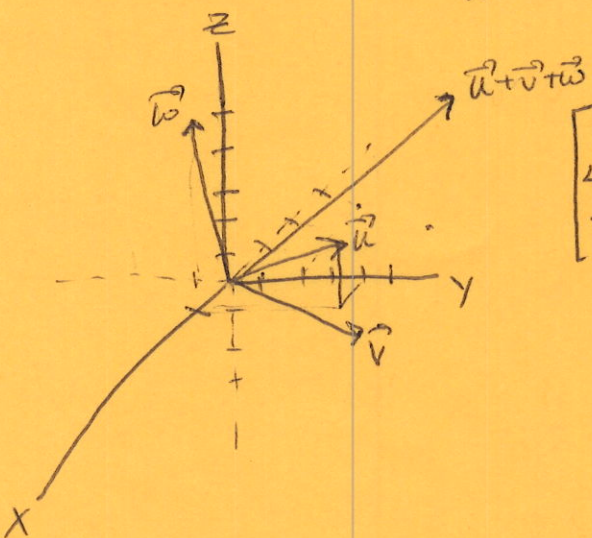
1.



$$\vec{u}+\vec{v}=\begin{bmatrix}1\\4\end{bmatrix}+\begin{bmatrix}-3\\2\end{bmatrix}=\begin{bmatrix}-2\\6\end{bmatrix}$$

if you imagine adding a copy of $\vec{v}$ to the end of $\vec{u}$ (and likewise a copy of $\vec{u}$ to the end of $\vec{v}$) we get a parallelogram and $\vec{u}+\vec{v}$ is the diagonal.

2.



$$\begin{bmatrix}1\\4\\2\end{bmatrix}+\begin{bmatrix}-3\\2\\-3\end{bmatrix}+\begin{bmatrix}0\\-1\\5\end{bmatrix}=\begin{bmatrix}-2\\5\\4\end{bmatrix}$$

3. a. $\vec{a}-\vec{b}=\begin{bmatrix}-1\\2\end{bmatrix}-\begin{bmatrix}3\\5\end{bmatrix}=\begin{bmatrix}-4\\-3\end{bmatrix}$

b. $3\vec{b}+2\vec{a}=3\begin{bmatrix}3\\5\end{bmatrix}+2\begin{bmatrix}-1\\2\end{bmatrix}=\begin{bmatrix}9\\15\end{bmatrix}+\begin{bmatrix}-2\\4\end{bmatrix}=\begin{bmatrix}7\\19\end{bmatrix}$

c. $\vec{c}+2\vec{d}-4\vec{e}=\begin{bmatrix}2\\7\\1\end{bmatrix}+2\begin{bmatrix}1\\-1\\0\end{bmatrix}-4\begin{bmatrix}4\\-5\\2\end{bmatrix}=\begin{bmatrix}2\\7\\1\end{bmatrix}+\begin{bmatrix}2\\-2\\0\end{bmatrix}+\begin{bmatrix}-16\\20\\-8\end{bmatrix}=\begin{bmatrix}-12\\25\\-7\end{bmatrix}$

4. a. false $f(t)=0$ for all t or its not the zero vector.

b. false it can be in higher dimensional spaces or not an "arrow".

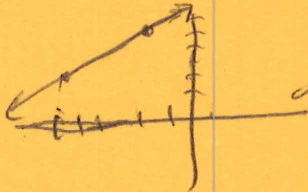
c. false. this is one condition, but not the only one

d. true

e. true

4f. false. R^2 is isomorphic to a subspace of R^3 but $\begin{bmatrix}a\\b\end{bmatrix}$ in R^2 is not in R^3.

g. true when $\vec{u}$ in H.

h. false  line does not go through origin

i. true

j. true

5. rref $\Rightarrow \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{array}\right] \Rightarrow 2\vec{v}_1 + \vec{v}_2 - 2\vec{v}_3 + \vec{v}_4 - \vec{v}_5$

the solution is unique.

6. a. this is not a subspace since b^2 (the second component) is positive and so it will fail scalar multiplication.

$-1\begin{bmatrix}1\\4\end{bmatrix} = \begin{bmatrix}-1\\-4\end{bmatrix}$ but $b^2 \neq -4$ for any real b.

this condition (b^2) is equivalent to saying $y \geq 0$. //

b. This is a subspace. i) if $a=b=0, c=0$ then $\begin{bmatrix}0\\0\\0\end{bmatrix}$ in V. ii) if $\begin{bmatrix}a\\b\\c\end{bmatrix}$ in V, and $\begin{bmatrix}e\\f\\g\end{bmatrix}$ in V then $\begin{bmatrix}a+e\\b+f\\c+g\end{bmatrix} = \begin{bmatrix}(b+c)+(f+g)\\b+f\\c+g\end{bmatrix} = \begin{bmatrix}(b+f)+(c+g)\\b+f\\c+g\end{bmatrix}$ in V.

$= \begin{bmatrix}(b+c)\\b\\c\end{bmatrix}$ $\quad = \begin{bmatrix}f+g\\f\\g\end{bmatrix}$ since it follows the definition. iii) for

$k\begin{bmatrix}a\\b\\c\end{bmatrix} = \begin{bmatrix}ka\\kb\\kc\end{bmatrix} = \begin{bmatrix}k(b+c)\\kb\\kc\end{bmatrix} = \begin{bmatrix}kb+kc\\kb\\kc\end{bmatrix}$ in V. So this is a subspace. //

c. this is not a subspace since $\vec{0}$ not in set. if $a=b=0$ we get $\begin{bmatrix}0 & 2\\0 & 0\end{bmatrix} \neq \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$. //

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6d. If you imagine the complex plane, $a+bi$ can be thought of as $\begin{bmatrix} a \\ b \end{bmatrix}$. this is the same as a vector in R^2, so yes, it is a vector space. we can show this is a subspace carefully. i) is zero in the space? yes. if $a=b=0$ then $0+0i=0$ ✓ ii) If I add two complex numbers is the result complex? yes, $(a+bi)+(c+di) = (a+c)+(b+d)i$ w/ $a+c$ and $b+d$ real. iii) can we scale & be in the space? yes. since k real means $k(a+bi) = (ka)+(kb)i$ w/ ka, kb real. //

6e. $p(t)$ divisible by $t-1$ means $p(t) = (t-1)q(t)$ where $q(t)$ is any polynomial. This is a subspace. if $q(t)=0$, then $(t-1)(0)$ $=0$. So 0 in the space. ii) if $s(t)$ & $p(t)$ in J, Then $s(t)+p(t) = (t-1)r(t) + (t-1)q(t) = (t-1)(r(t)+q(t))$. Since $r(t)$, $q(t)$ are polynomials, so is $(r(t)+q(t))$, so $s(t)+p(t)$ in J. iii) if k is real then $kp(t) = k(t-1)q(t) = (t-1)(kq(t))$ but $kq(t)$ is a polynomial so $kp(t)$ in J. //

f. yes, this is a subspace. i) $f(x)=0$ then $f(-x)=0$ so 0 is an odd function since $f(-x)=f(x)=-f(x)$. ii) if f and g are both odd then $f(x)+g(x) = -f(-x)+(-g(-x)) = -(f(-x)+g(-x))$ and so $-(f(x)+g(x)) = f(-x)+g(-x)$. So the sum of two odd functions is odd. iii) and $kf(x)$ is odd, then $kf(-x) = k(-f(x)) = -kf(x)$. which is also odd. //

g. all $A^2 = A$ $(n\times n)$. i) The zero matrix is in the space since $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. ii) if A, B in W, the $A^2=A$ and $B^2=B$. $(A+B)^2 = A+B$? $A^2+AB+BA+B^2 = A+B$? this is not true for all A, B in W since

6g cont'd I in W and if $A=I$, then $A^2+AB+BA+B^2 =$
$(A^2+B^2)+2B \neq A+B$ except in the special case when $B=0$.
This is not a subspace. //

h. exponential functions are typically defined a^x for $a>0$. So there is no zero in the space. Even if we allow a^x for $a=0$, a^x+b^x is not exponential. //

i. The set of all singular matrices does contain the zero matrix (since the zero matrix is singular). But $\begin{bmatrix}1&0\\0&0\end{bmatrix}$ and $\begin{bmatrix}0&0\\0&1\end{bmatrix}$ are singular but $\begin{bmatrix}1&0\\0&0\end{bmatrix}+\begin{bmatrix}0&0\\0&1\end{bmatrix}=\begin{bmatrix}1&0\\0&1\end{bmatrix}$ which is nonsingular. //

7a. $\begin{bmatrix}x_1\\y_1\end{bmatrix}+\begin{bmatrix}x_2\\y_2\end{bmatrix}=\begin{bmatrix}x_1+x_2\\y_1+y_2\end{bmatrix}$, $c\begin{bmatrix}x_1\\y_1\end{bmatrix}=\begin{bmatrix}\sqrt{c}\,x_1\\\sqrt{c}\,y_1\end{bmatrix}$

the definition of addition here is the same as the normal definition, so they should all still work. What we need to check are scalar multiplication properties. We immediately run into problems w/ $c\vec{u}$ in V. If $c<0$ then $\sqrt{c}$ is not real, so this is not a vector space of real numbers any longer or is not defined. Further $-1\vec{u}=\sqrt{-1}\,\vec{u} \neq$ to the opposite of $\vec{u}$ since $\vec{u}+(-1\vec{u})\neq\vec{0}$.
Therefore, this definition of scalar multiplication does not produce a vector space.

b. $\begin{bmatrix}x_1\\y_1\\z_1\end{bmatrix}+\begin{bmatrix}x_2\\y_2\\z_2\end{bmatrix}=\begin{bmatrix}x_1+x_2+1\\y_1+y_2+1\\z_1+z_2+1\end{bmatrix}$, $c\begin{bmatrix}x_1\\y_1\\z_1\end{bmatrix}=\begin{bmatrix}cx_1+c-1\\cy_1+c-1\\cz_1+c-1\end{bmatrix}$

both definitions have changed, so I'll begin w/ property i.

i) $\vec{u}+\vec{v}$ in V if $\vec{u}=\begin{bmatrix}x_1\\y_1\\z_1\end{bmatrix}$ and $\vec{v}=\begin{bmatrix}x_2\\y_2\\z_2\end{bmatrix}$ then $\vec{u}+\vec{v}=\begin{bmatrix}x_1+x_2+1\\y_1+y_2+1\\z_1+z_2+1\end{bmatrix}$ is okay since these entries are still real, 7b cont'd.

ii) $\vec{u}+\vec{v} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} x_1+x_2+1 \\ y_1+y_2+1 \\ z_1+z_2+1 \end{bmatrix}$ and $\vec{v}+\vec{u} = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} + \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} =$

$\begin{bmatrix} x_2+x_1+1 \\ y_2+y_1+1 \\ z_2+z_1+1 \end{bmatrix}$ Since each entry is made up of real numbers, these are the same by commutativity.

iii) $\vec{u}+(\vec{v}+\vec{w}) = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + \left(\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} + \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix} \right) = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + \begin{bmatrix} x_2+x_3+1 \\ y_2+y_3+1 \\ z_2+z_3+1 \end{bmatrix} =$

$\begin{bmatrix} x_1+(x_2+x_3+1)+1 \\ y_1+(y_2+y_3+1)+1 \\ z_1+(z_2+z_3+1)+1 \end{bmatrix} = \begin{bmatrix} x_1+x_2+x_3+2 \\ y_1+y_2+y_3+2 \\ z_1+z_2+z_3+2 \end{bmatrix}$

and $(\vec{u}+\vec{v})+\vec{w} = \left(\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} \right) + \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix} = \begin{bmatrix} x_1+x_2+1 \\ y_1+y_2+1 \\ z_1+z_2+1 \end{bmatrix} + \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix}$

$= \begin{bmatrix} (x_1+x_2+1)+x_3+1 \\ (y_1+y_2+1)+y_3+1 \\ (z_1+z_2+1)+z_3+1 \end{bmatrix} = \begin{bmatrix} x_1+x_2+x_3+2 \\ y_1+y_2+y_3+2 \\ z_1+z_2+z_3+2 \end{bmatrix}$

these are the same by properties of real #'s, so this property holds.

iv) $\vec{u} + \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \vec{u}$? does such a vector exist?

$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} x_1+a+1 \\ y_1+b+1 \\ z_1+c+1 \end{bmatrix} \stackrel{?}{=} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ This suggests that the vector $\begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}$ acts like $\vec{0}$ vector (the additive identity). This may cause problems.

v). $\vec{u} + \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}$ what vector acts like the additive inverse?

$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} x_1+a+1 \\ y_1+b+1 \\ z_1+c+1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix} \Rightarrow \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -x_1-2 \\ -y_1-2 \\ -z_1-2 \end{bmatrix}$

does this satisfy our definition of $-\vec{u}$?

$-1 \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} -x_1+(-1)-1 \\ -y_1+(-1)-1 \\ -z_1+(-1)-1 \end{bmatrix} = \begin{bmatrix} -x_1-2 \\ -y_1-2 \\ -z_1-2 \end{bmatrix}$. it does, so the property checks out so far.

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7b cont'd.

vi) $c\vec{u} = \begin{bmatrix} cx_1 + c - 1 \\ cy_1 + c - 1 \\ cz_1 + c - 1 \end{bmatrix}$ these entries are real numbers, and so $c\vec{u}$ is in V.

vii) $\vec{c}(\vec{u}+\vec{v}) = c\vec{u} + c\vec{v}$?

$$c\left(\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}\right) = c\begin{bmatrix} x_1 + x_2 + 1 \\ y_1 + y_2 + 1 \\ z_1 + z_2 + 1 \end{bmatrix} = \begin{bmatrix} c(x_1 + x_2 + 1) + c - 1 \\ c(y_1 + y_2 + 1) + c - 1 \\ c(z_1 + z_2 + 1) + c - 1 \end{bmatrix} =$$

$$\begin{bmatrix} cx_1 + cx_2 + c + c - 1 \\ cy_1 + cy_2 + c + c - 1 \\ cz_1 + cz_2 + c + c - 1 \end{bmatrix} = \begin{bmatrix} cx_1 + cx_2 + 2c - 1 \\ cy_1 + cy_2 + 2c - 1 \\ cz_1 + cz_2 + 2c - 1 \end{bmatrix}$$

$$c\vec{u} + c\vec{v} = c\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + c\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} cx_1 + c - 1 \\ cy_1 + c - 1 \\ cz_1 + c - 1 \end{bmatrix} + \begin{bmatrix} cx_2 + c - 1 \\ cy_2 + c - 1 \\ cz_2 + c - 1 \end{bmatrix}$$

$$= \begin{bmatrix} cx_1 + cx_2 + c + c - 2 \\ cy_1 + cy_2 + c + c - 2 \\ cz_1 + cz_2 + c + c - 2 \end{bmatrix} = \begin{bmatrix} cx_1 + cx_2 + 2c - 2 \\ cy_1 + cy_2 + 2c - 2 \\ cz_1 + cz_2 + 2c - 2 \end{bmatrix}$$

these are not equal, so the property fails.

the set V with these operations is not a vector space.

7c. $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 x_2 \\ y_1 y_2 \end{bmatrix}$, $c\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} cx_1 \\ cy_1 \end{bmatrix}$

i) $\vec{u} + \vec{v} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 x_2 \\ y_1 y_2 \end{bmatrix}$ these entries are real numbers so they are in V.

ii) $\vec{v} + \vec{u} = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} + \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} x_2 x_1 \\ y_2 y_1 \end{bmatrix}$ since multiplication is commutative on real numbers

this is the same as $\vec{u} + \vec{v}$.

iii) $\vec{u} + (\vec{v} + \vec{w}) = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \left(\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} + \begin{bmatrix} x_3 \\ y_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 x_3 \\ y_2 y_3 \end{bmatrix} = \begin{bmatrix} x_1 x_2 x_3 \\ y_1 y_2 y_3 \end{bmatrix}$

and $(\vec{u} + \vec{v}) + \vec{w} = \left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right) + \begin{bmatrix} x_3 \\ y_3 \end{bmatrix} = \begin{bmatrix} x_1 x_2 \\ y_1 y_2 \end{bmatrix} + \begin{bmatrix} x_3 \\ y_3 \end{bmatrix} = \begin{bmatrix} x_1 x_2 x_3 \\ y_1 y_2 y_3 \end{bmatrix}$

These are equal.

7c. cont'd.

iv) $\vec{u}+\vec{0}=\vec{u}$ what is $\vec{0}$? $\begin{bmatrix}x_1\\y_1\end{bmatrix}+\begin{bmatrix}a\\b\end{bmatrix}\stackrel{?}{=}\begin{bmatrix}x_1\\y_1\end{bmatrix} \Rightarrow \begin{bmatrix}x_1\cdot a\\y_1\cdot b\end{bmatrix}=\begin{bmatrix}x_1\\y_1\end{bmatrix}$

So $\vec{0}=\begin{bmatrix}1\\1\end{bmatrix}$.

v) $-\vec{u}$? $\vec{u}+\begin{bmatrix}a\\b\end{bmatrix}=\begin{bmatrix}1\\1\end{bmatrix} \Rightarrow \begin{bmatrix}x_1\\y_1\end{bmatrix}+\begin{bmatrix}a\\b\end{bmatrix}=\begin{bmatrix}1\\1\end{bmatrix} \Rightarrow \begin{bmatrix}x_1a\\y_1b\end{bmatrix}=\begin{bmatrix}1\\1\end{bmatrix}$

$\Rightarrow \begin{matrix}a=1/x_1\\b=1/y_1\end{matrix}$ $\quad -\vec{u}=\begin{bmatrix}1/x_1\\1/y_1\end{bmatrix}$ which exists if neither x_1 nor y_1 is zero.

this is actually a problem since addition requires an additive inverse for all reals including 0.

further this is not the same as $-1\vec{u}=\begin{bmatrix}-x_1\\-y_1\end{bmatrix}$. So this is not a vector space.

7d. $x+y=xy$, $cx=x^c$ in R^+ (there is no zero in this set - and all values are positive)

i) $x+y=xy$ This is in the set since if $x>0, y>0$ then $xy>0$.

ii) $x+y=xy$ and $y+x=yx$. Since multiplication on real numbers is commutative, these are equal.

iii) $\vec{u}+(\vec{v}+\vec{w})=\vec{u}+vw \doteq u(vw)$ and $(u+v)+w=(uv)+w=(uv)w$ multiplication on reals is associative, so these are equal.

iv) $x+y=x \Rightarrow xy=x$ so $y=1$ $\quad \vec{0}=1$.

v) $-\vec{u}+\vec{u}=0$? $x+y=1$ since $\vec{0}=1 \Rightarrow xy=1 \Rightarrow y=1/x$.

this is consistent w/ definition of $-x=x^{-1}=\frac{1}{x}$.

vi) $cx=x^c$ This is a positive number for every positive x and any real c, so this is in the set.

7d cont'd.

vii) $c(x+y) = c(xy) = (xy)^c$

and $cx + cy = x^c y^c$ these are equal.

viii) $(c+d)x = x^{c+d}$ and $cx + dx = x^c + x^d = x^c x^d = x^{c+d}$

These are equal.

ix) $= (cd)x = x^{cd}$ and $c(dx) = c(x^d) = (x^d)^c = x^{cd}$

these are equal.

x. $1x = x^1 = x$ also works.

So this does satisfy all the properties of a vector space.