MAT 201, Exam #1, Fall 2016 Name KEY

Instructions: Show all work. Answers without work required to obtain the solution will not receive full credit. Some questions may contain multiple parts: be sure to answer all of them. Give exact answers unless specifically asked to estimate.

1. Fin d the distance from the point $(4, -2, 1)$ to: (3 points each)
   a. the line $\vec{r}(t) = (3t-2)\hat{\imath} + (4-t)\hat{\jmath} + (2t+1)\hat{k}$

$Q(-2,4,1)$

$\vec{PQ} = \langle -6, 6, 0 \rangle$

$\vec{u} = \langle 3, -1, 2 \rangle$

$$\vec{PQ} \times \vec{u} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -6 & 6 & 0 \\ 3 & -1 & 2 \end{vmatrix} = (12-0)\hat{\imath} - (-12-0)\hat{\jmath} + (6-18)\hat{k}$$

$$\|\vec{PQ} \times \vec{u}\| = \sqrt{144+144+144} = 12\sqrt{3}$$

$$\|\vec{u}\| = \sqrt{9+1+4} = \sqrt{14} \qquad d = \frac{12\sqrt{3}}{\sqrt{14}}$$

   b. the plane $6x + 3y + z = 12$

$Q\ (0,0,12)$

$\vec{PQ} = \langle 4, -2, -11 \rangle$

$\vec{n} = \langle 6, 3, 1 \rangle$

$$\vec{PQ} \cdot \vec{n} = 24 - 6 - 11 = 7$$

$$\|\vec{n}\| = \sqrt{36+9+1} = \sqrt{46}$$

$$d = \frac{7}{\sqrt{46}}$$

2. Find an equation of the sphere with endpoints of a diameter at $(2,1,4)$ and $(4,3,10)$. (3 points)

$$\text{center} = \left(\frac{2+4}{2}, \frac{1+3}{2}, \frac{4+10}{2}\right) = (3,2,7)$$

$$r = \sqrt{1^2+1^2+3^2} = \sqrt{11}$$

$$(x-3)^2 + (y-2)^2 + (z-7)^2 = 11$$

3. For the vectors $\vec{u} = -3\hat{\imath} + 7\hat{\jmath} + \hat{k}, \vec{v} = \langle 1, 3, -2 \rangle$, find the following: (2 points each)
   a. $\vec{u} + 3\vec{v}$

$-3 + 3(1) = 0$

$7 + 3(3) = 7 + 9 = 16$

$1 + 3(-2) = 1 - 6 = -5$

$$\vec{u} + 3\vec{v} = \langle 0, 16, -5 \rangle$$

b. $||\vec{u}||$

$$\sqrt{9+49+1} = \sqrt{59}$$

c. A unit vector in the direction of $\vec{v}$

$$\sqrt{1+9+4} = \sqrt{14} \qquad \hat{v} = \left\langle 1/\sqrt{14},\ 3/\sqrt{14},\ -2/\sqrt{14} \right\rangle$$

d. $\vec{u} \times \vec{v}$

$$\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -3 & 7 & 1 \\ 1 & 3 & -2 \end{vmatrix} = (-14-3)\hat{\imath} - (6-1)\hat{\jmath} + (-9-7)\hat{k}$$

$$-17\hat{\imath} - 5\hat{\jmath} - 16\hat{k}$$

e. $\vec{u} \cdot \vec{v}$

$$-3+21-2 = 16$$

f. The angle between $\vec{u}$ and $\vec{v}$

$$\cos\theta = \frac{16}{\sqrt{59}\sqrt{14}} \qquad \theta = .98037\ldots$$

$$\approx 56^\circ$$

4. Find the volume of the parallelepiped determined by the vectors $\langle 4,2,2\rangle$, $\langle 3,3,-1\rangle$, $\langle 5,5,1\rangle$. (3 points)

$$\begin{vmatrix} 4 & 2 & 2 \\ 3 & 3 & -1 \\ 5 & 5 & 1 \end{vmatrix} = 4(3+5) - 2(3+5) + \overset{0}{\cancel{2(18-15)}}$$

$$4(8) - 2(8) = 2(8) = \boxed{16}$$

5. Consider the vector $\vec{w} = \hat{\imath} + \sqrt{3}\hat{\jmath}$. Find the magnitude and angle between the vector and the positive x-axis. Use that information to write the vector in polar form. (3 points)

$\|\vec{w}\| = \sqrt{1+3} = \sqrt{4} = 2$

$\tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = \pi/3 = \theta$

$\vec{w} = 2\cos \pi/3 \, \hat{\imath} + 2\sin \pi/3 \, \hat{\jmath}$

6. Find an equation of the line through $(1, -1, 1)$, parallel to $x + 2 = \frac{1}{2}y = z - 3$. Write your equation in vector-valued form. (3 points)

$\langle 1, 2, 1 \rangle$

$\vec{r}(t) = (t+1)\hat{\imath} + (2t-1)\hat{\jmath} + (t+1)\hat{k}$

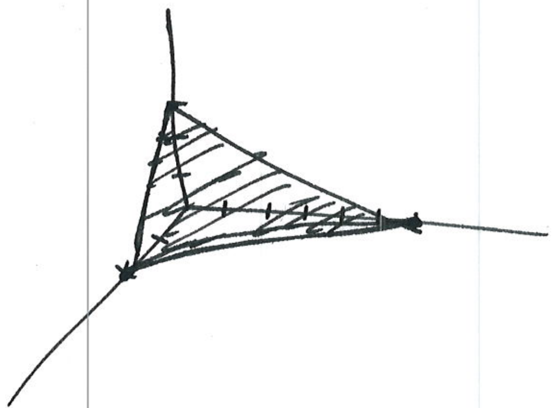
7. Find an equation of the plane containing the point $(2,0,1)$ and parallel to the plane $5x + 2y + z = 1$. (3 points)

$\langle 5, 2, 1 \rangle$

$5(x-2) + 2(y-0) + 1(z-1) = 0$

8. Use intercepts to graph the plane $3x + y + 2z = 6$. (3 points)

(2,0,0)
(0,6,0)
(0,0,3)



9. Identify the quadric surface, sketch or describe the graph including the orientation. (2 points each)

a. $9x^2 + 4y^2 + z^2 = 1$

ellipsoid → longest stretch in z direction

b. $y^2 = x^2 + 2z^2$

elliptical cone wrapped around y-axis

c. $x^2 - y^2 + z^2 - 4x - 2y - 4z + 4 = 0$

$(x^2-4x+4)-(y^2+2y+1)+(z^2-4z+4) = -\cancel{4}+\cancel{4}-1+4$

hyperboloid of one sheet wrapped around y axis

d. $y = x^2 - z^2$

hyperbolic paraboloid $\perp$ to y-axis

e. $y^2 = x^2 + 4z^2 + 4$

hyperboloid of 2 sheets.

10. Rewrite the equation $z = xy$ in (3 points each)

a. Cylindrical coordinates

$$z = r^2 \cos\theta \sin\theta$$

b. Spherical coordinates

$$\rho \cos\varphi = \rho^2 \sin^2\varphi \cos\theta \sin\theta$$

$$\rho = \cot\varphi \csc\varphi \sec\theta \csc\theta$$

11. Rewrite the equations in rectangular coordinates. (3 points each)

a. $z = 2r$

$$z = 2\sqrt{x^2+y^2} \quad \text{or} \quad \frac{z^2}{4} = x^2+y^2$$

b. $\rho = 4\cos\phi$

$$\rho^2 = 4\rho\cos\varphi$$

$$x^2+y^2+z^2 = 4z$$

12. Find a parametric equation that represents the intersection of $x^2 + y^2 = 4$ and $z = xy$. (5 points)

$$x = 2\cos\theta = 2\cos t$$

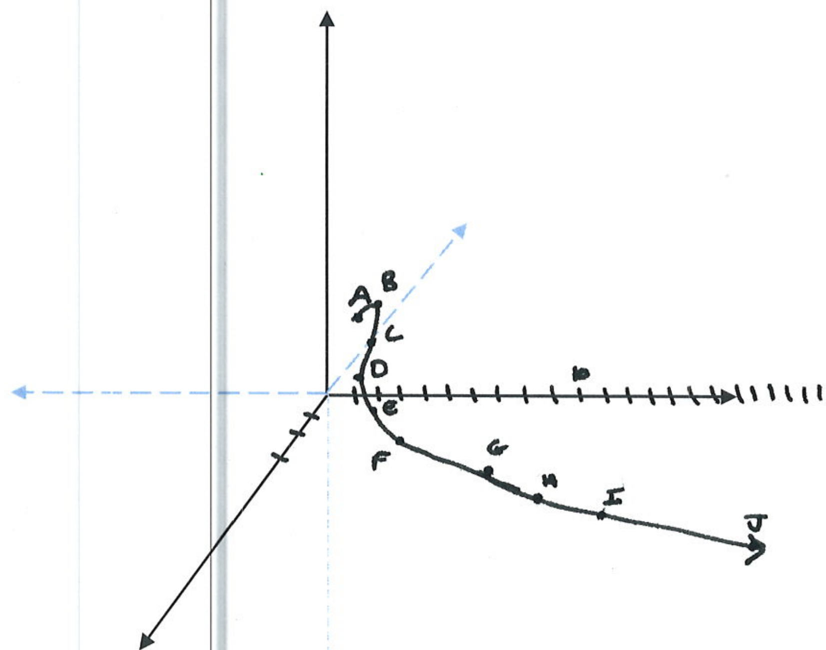
$$y = 2\sin t$$

$$z = 4\cos t \sin t$$

$$\vec{r}(t) = 2\cos t\,\hat{i} + 2\sin t\,\hat{j} + 4\cos t\sin t\,\hat{k}$$

13. Sketch the graph of $\vec{r}(t) = t\hat{\imath} + e^t\hat{\jmath} + \cos t\,\hat{k}$. Plot at around 10 points. (10 points)

t	x	y	z	
0	0	1	1	B
$\pi/6$	$\pi/6$	≈ 1.69	$.866$	C
$\pi/4$	$\pi/4$	≈ 2.19	$.7$	D
$\pi/3$	$\pi/3$	≈ 2.85	$.5$	E
$\pi/2$	$\pi/2$	≈ 4.81	0	F
$2\pi/3$	$2\pi/3$	≈ 8.12	$-.5$	G
$3\pi/4$	$3\pi/4$	≈ 10.55	$-.71$	H
$5\pi/6$	$5\pi/6$	≈ 13.71	$-.86$	I
π	π	≈ 23.14	-1	J
$-\pi/6$	$-\pi/6$	$\approx .59$	$.866$	A



14. Find the domain and range of the function $f(x,y) = x + \ln(12 - 2x - 3y)$ write in proper set notation. (5 points)

$12 - 2x - 3y > 0$

$12 > 2x + 3y$

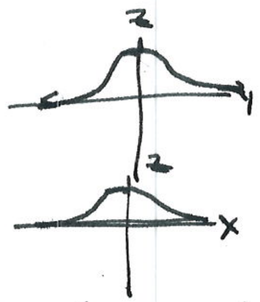
D: $\{(x,y) \mid 2x + 3y < 12\}$

R: $(-\infty, \infty)$

15. Draw the trace of the graph $f(x,y) = \frac{1}{1+x^2+y^2}$ on the plane $x = 0$ and $y = 0$. What do the level curves look like? (6 points)

$x = 0 \Rightarrow z = \frac{1}{1+y^2}$

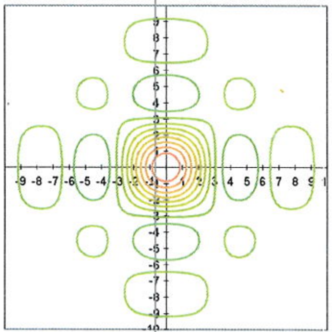
$y = 0 \Rightarrow z = \frac{1}{1+x^2}$

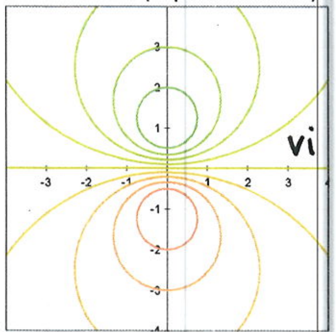


level curves are circles for $z \in (0, 1]$

16. Match each function to its set of level curves. (2 points each)

iv

a. 

c. 

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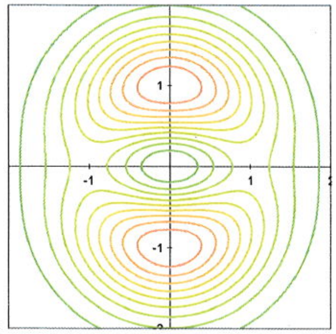
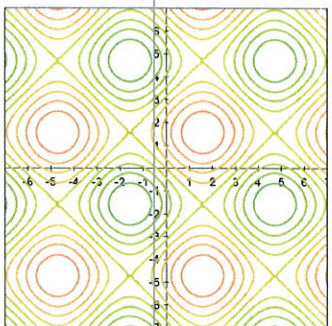
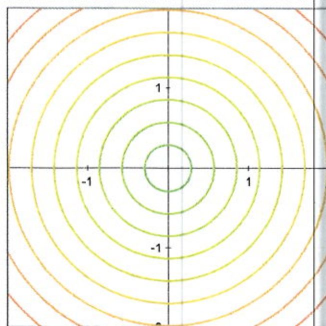
e. 

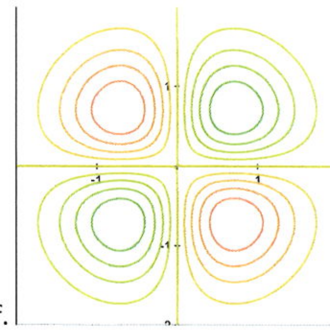
i

ii

b. 

d. 

iii

f. 

v

i. $z = (x^2 + 3y^2)e^{-x^2-y^2}$ E

ii. $z = \sin x + \sin y$ B

iii. $z = \sqrt{x^2 + y^2}$ D

iv. $z = \frac{\sin x \sin y}{xy}$ A

v. $z = xye^{-x^2-y^2}$ F

vi. $z = -\frac{3y}{x^2+y^2+1}$ C

17. Find the limit. (5 points each)

a. $\lim_{(x,y,z)\to(0,0,0)} \frac{xy+yz}{x^2+y^2+z^2}$

$$\lim_{\rho\to 0} \frac{\cancel{\rho^2}\sin^2\varphi\cos\theta\sin\theta + \cancel{\rho^2}\sin\varphi\sin\theta\cos\varphi}{\cancel{\rho^2}}$$

$$= \lim_{\rho\to 0} \sin^2\varphi\cos\theta\sin\theta + \sin\varphi\sin\theta\cos\varphi$$

$$\neq 0$$

Since value depends on path

DNE

b. $\lim_{(x,y)\to(0,0)} \frac{xy^4}{x^2+y^8}$

$x = ky^4$

$$\lim_{y\to 0} \frac{ky^4y^4}{k^2y^8+y^8} = \lim_{y\to 0} \frac{k\cancel{y^8}}{\cancel{y^8}(k^2+1)}$$

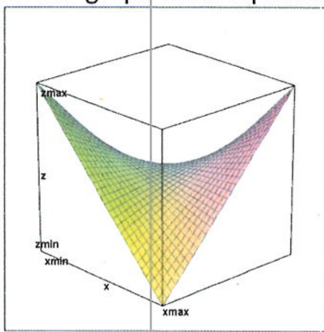
$$= \frac{k}{k^2+1}$$

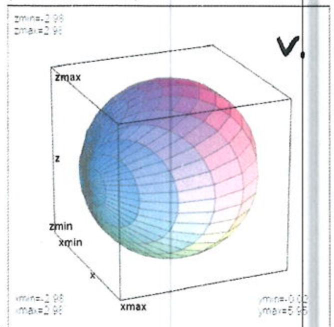
Since this value depends on path

DNE

18. Match the graph to its equation. (2 point each)

ii

a. 

c. 

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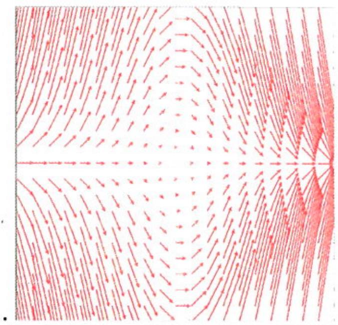
e. 

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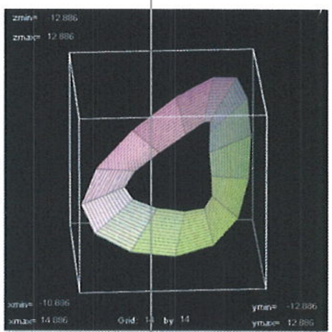
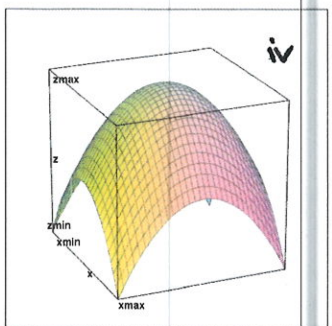
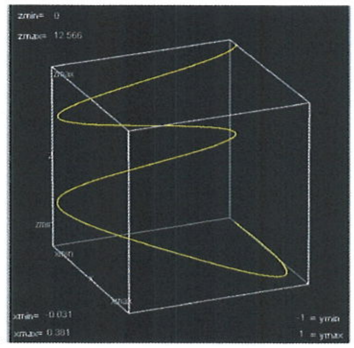
vi

b. 

d. 

iv

f. 

iii

i. $\vec{F}(x,y) = \sqrt{x^2+y^2}\,\hat{\imath} - xy\hat{\jmath}$ E

ii. $z = xy$ A

iii. $\vec{r}(t) = e^{-0.8t}\sin t\,\hat{\imath} + \cos t\,\hat{\jmath} + t\hat{k}$ F

iv. $z = 4 - r^2$ D

v. $\rho = 6\sin\theta\sin\phi$ C

vi. $\vec{r}(u,v) = \left(2\cos u + v\cos\left(\frac{u}{2}\right)\right)\hat{\imath} + \left(2\sin u + v\cos\left(\frac{u}{2}\right)\right)\hat{\jmath} + v\sin\left(\frac{u}{2}\right)\hat{k}$ B

19. Find a parametric equation for the part of the cylinder $y^2 + z^2 = 16$ between $x = 0, x = 5$. Be sure to state the bounds on the variables. (6 points)

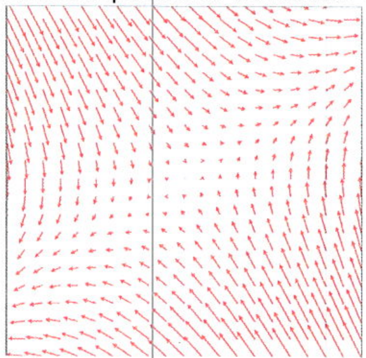
$\vec{r}(u,v) =$

$v\hat{\imath} + 4\cos u\,\hat{\jmath} + 4\sin u\,\hat{k}$

$0 \le u \le 2\pi, \quad 0 \le v \le 5$

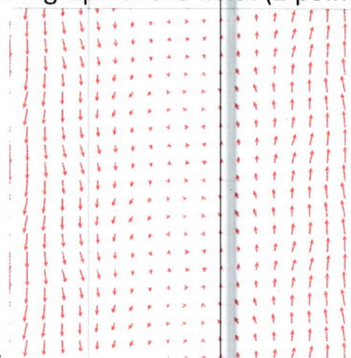
20. Match the equation of the vector field to the graph of the field. (2 points each)

ii



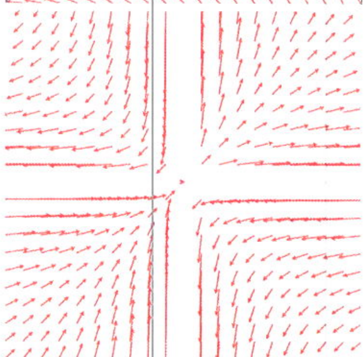
a.

iv



c.

iii



b.

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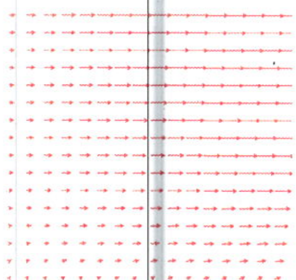

d.

i. $\vec{F}(x,y) = \sqrt{xy}\hat{\imath} + e^{-y}\hat{\jmath}$ D

ii. $\vec{F}(x,y) = y\hat{\imath} + (x-y)\hat{\jmath}$ A

iii. $\vec{F}(x,y) = \frac{x}{y}\hat{\imath} + \frac{y}{x}\hat{\jmath}$ B

iv. $\vec{F}(x,y) = (\cos x + \sin y)\hat{\imath} + x\hat{\jmath}$ C

Some useful formulas:

$$d = \frac{|\overrightarrow{PQ} \cdot \vec{n}|}{||\vec{n}||}$$

$$d = \frac{||\overrightarrow{PQ} \times \vec{u}||}{||\vec{u}||}$$