McCall

Math 2568, Exam #1-Part I, Fall 2014 Name KEY

Instructions: You may **not** use a calculator on this portion of the exam. You should show all work and use exact answers.

1. The system shown below is in vector equation form.

$$x_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ -8 \end{bmatrix}$$

a. Write the equation as a system of linear equations in two variables. (3 points)

$$\begin{cases} x_1 + 3x_2 = -3 \\ -2x_1 + x_2 = -8 \end{cases}$$

b. Write the system as an augmented matrix. (3 points)

$$\left[\begin{array}{cc|c} 1 & 3 & -3 \\ -2 & 1 & -8 \end{array}\right]$$

c. Write the system as a matrix equation. (3 points)

$$\begin{bmatrix} 1 & 3 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -3 \\ -8 \end{bmatrix}$$

d. Solve the system, by reducing the matrix with row operations. Write the solution as a column vector. (6 points)

$2R_1 + R_2 \to R_2$

$$\left[\begin{array}{cc|c} 1 & 3 & -3 \\ 0 & 7 & -14 \end{array}\right]$$

$\frac{1}{7}R_2 \to R_2$

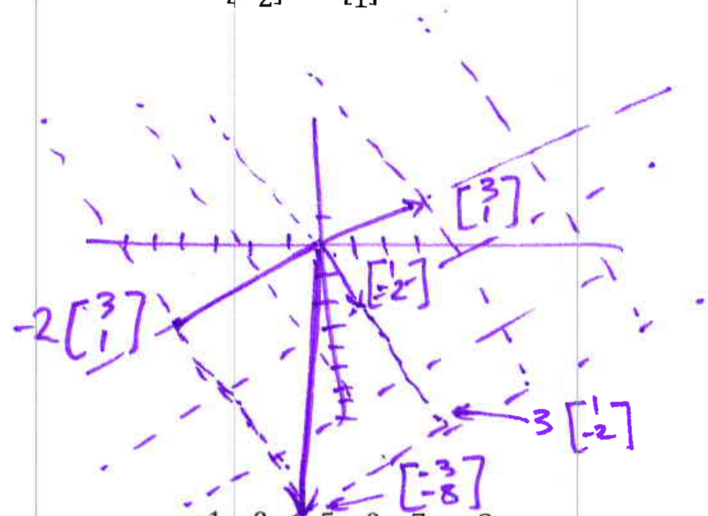
$$\left[\begin{array}{cc|c} 1 & 3 & -3 \\ 0 & 1 & -2 \end{array}\right]$$

$-3R_2 + R_1 \to R_1$

$$\left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -2 \end{array}\right]$$

$x_1 = 3$, $x_2 = -2 \Rightarrow \vec{x} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$

e. Use the solution you obtained and graphically represent it on a graph as the linear combination of the vectors $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$. (8 points)



2. Suppose that the matrix $A = \begin{bmatrix} 1 & 0 & -5 & 0 & 7 & -3 \\ 0 & 1 & 3 & 1 & 0 & 4 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ is the coefficient matrix of a homogeneous system that is already partially reduced. Finish reducing the system, and then state the solution in parametric form. (8 points)

$-7R_3 + R_1 \to R_1$

free

$$\begin{bmatrix} 1 & 0 & -5 & 0 & 0 & 4 \\ 0 & 1 & 3 & 1 & 0 & 4 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

pivot pivot pivot

$x_1 = 5x_3 - 4x_6$

$x_2 = -3x_3 - x_4 - 4x_6$

$x_3 = x_3$

$x_4 = x_4$

$x_5 = x_6$

$x_6 = x_6$

$$\vec{x} = \begin{bmatrix} 5 \\ -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} x_3 + \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} x_4 + \begin{bmatrix} -4 \\ -4 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} x_6$$

3. Determine if each statement is True or False. (1 point each)

a. T F Every elementary row operation is irreversible.

b. T F Two equivalent linear systems can have different solutions.

c. T F The reduced echelon form of a matrix is unique.

d. T F A homogeneous system with free variables has only the trivial solution.

e. T F If vector $\vec{x}$ is a linear combination of vectors $\vec{v_i}$, then $\vec{x}$ is in the span of $\{\vec{v_1}, \dots \vec{v_p}\}$.

f. T F Any set of five real numbers can be represented as a vector in $\mathbb{R}^4$. on $\mathbb{R}^5$ okay

g. T F If A has a pivot in every row, then $A\vec{x} = \vec{b}$ has a solution for every $\vec{b}$ in $\mathbb{R}^m$.

h. T F Both $\begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & * & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 \end{bmatrix}$ are matrices in echelon form.

i. T F If T is a linear transformation mapping $\mathbb{R}^4 \mapsto \mathbb{R}^5$, then it can be represented by a 5x4 matrix.

j. T F If $\{\vec{u}, \vec{v}, \vec{x}, \vec{z}\}$ is linearly independent, then so is $\left\{\vec{u}, \vec{v}, \vec{w}, \vec{x}, \vec{y}, \vec{z}\right\}$.

k. T F A homogeneous equation is always inconsistent.

l. T F $\{\vec{0}\}$ is a subspace.

m. T F A linear transformation defined by a 5x6 matrix can be onto, but it cannot be one-to-one.

n. T F If two spaces have the same number of basis vectors, then then are isomorphic.

o. T F The pivot rows of a matrix are always linearly dependent.

p. T F The null space of an mxn matrix is a subspace of R^m.

q. T F If A and B are row equivalent, then their column spaces are the same. same dimension only

r. T F The vector space $\mathbb{P}_{n+1}$ and R^n are isomorphic. $R^{n+1} \approx P_n$

s. T F A linearly independent set in a subspace H that spans the space is a basis for H.

t. T F The kernel of a matrix is a subspace of the domain of the matrix.

u. T F An isomorphism is a mapping that is both one-to-one and onto.

v. T F The third standard basis vector $\vec{e_4}$ in P_6 is t^3.

t^3 is e_4 but it's the 4th basis vector

4. Write a matrix D with entries $d_{ij} = \begin{cases} i+j, \ for\ i>j \\ 0, \ for\ i=j \\ i-j, \ for\ i<j \end{cases}$. (5 points)

ex.

$$\begin{bmatrix} 0 & -1 & -2 & -3 \\ 3 & 0 & -1 & -2 \\ 4 & 5 & 0 & -1 \\ 5 & 6 & 7 & 0 \end{bmatrix} = \begin{bmatrix} d_{11} & d_{12} & d_{13} & d_{14} \\ d_{21} & d_{22} & d_{23} & d_{24} \\ d_{31} & d_{32} & d_{33} & d_{34} \\ d_{41} & d_{42} & d_{43} & d_{44} \end{bmatrix}$$

5. Define the following terms as completely as possible. (4 points each)
   a. What does it mean for a linear transformation to be **onto**?

a linear transform is onto if There is always a solution to $A\vec{x}=\vec{b}$ for any $\vec{b}$ in $\mathbb{R}^m$ (the codomain)

   b. What is a **projection** transformation? Give an example of it.

a projection transformation collapses at least one dimension of the space to zero. typically $\mathbb{R}^n \Rightarrow$ a subspace of $\mathbb{R}^n$

i.e. $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ takes any $\begin{bmatrix} x \\ y \end{bmatrix}$ & maps onto only $\begin{bmatrix} x \\ 0 \end{bmatrix}$

6. Perform the indicated row operations on the matrix $\begin{bmatrix} 2 & 4 & 1 \\ 2 & -4 & -4 \\ 5 & 0 & -1 \end{bmatrix}$. Base each operation on the original matrix, not in succession. (9 points)

a. Add$(-1)R_1 + R_2$ $\to R_2$

$$\begin{bmatrix} 2 & 4 & 1 \\ 0 & -8 & -5 \\ 5 & 0 & -1 \end{bmatrix}$$

b. Multiply $\frac{1}{2}R_2$ $\to R_2$

$$\begin{bmatrix} 2 & 4 & 1 \\ 1 & -2 & -2 \\ 5 & 0 & -1 \end{bmatrix}$$

c. Exchange $R_1 \leftrightarrow R_3$

$$\begin{bmatrix} 5 & 0 & -1 \\ 2 & -4 & -4 \\ 2 & 4 & 1 \end{bmatrix}$$

7. Determine if the transformation $T: M_{2\times2} \to M_{2\times2}, T\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ad & 0 \\ bc & 0 \end{bmatrix}$ is linear or not. If it is, prove it. If it is not, give a counterexample. (6 points)

it is not a linear transformation

$$2T\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right) = 2\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 2 & 0 \end{bmatrix}$$

$$\text{but } T\left(2\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right) = T\left(\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}\right) = \begin{bmatrix} 4 & 0 \\ 4 & 0 \end{bmatrix}$$

Should also fail addition test.
$T(\vec{0})$ test checks out.

Math 2568, Exam #1-Part II, Fall 2014 Name KEY

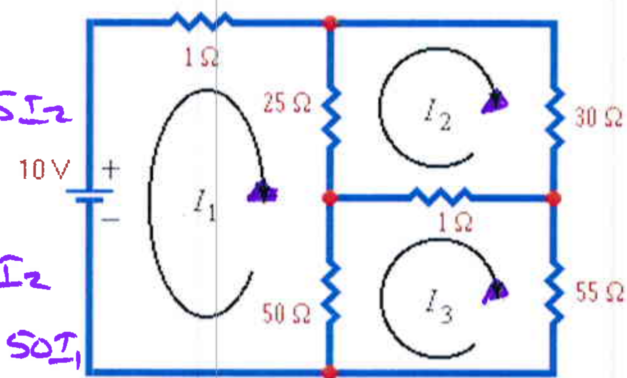
Instructions: You may use a calculator on this portion of the exam, however, I cannot award partial credit where you show no work, and all answers must be justified in some fashion. You should show reduced matrices obtained from the calculator together with their interpretation where appropriate.

1. Based on the graph below, solve the system for the circuit flowing through each loop. The resistance is in Ohms. (7 points)

$$-10 = 50I_1 - 50I_3 + 25I_1 - 25I_2 + I_1$$

$$0 = 25I_2 - 25I_1 + I_2 - I_3 + 30I_2$$

$$0 = 55I_3 + I_3 - I_2 + 50I_3 - 50I_1$$



$$76I_1 - 25I_2 - 50I_3 = -10$$
$$-25I_1 + 56I_2 - I_3 = 0$$
$$-50I_1 - I_2 + 106I_3 = 0$$

$$\left[\begin{array}{ccc|c} 76 & -25 & -50 & -10 \\ -25 & 56 & -1 & 0 \\ -50 & -1 & 106 & 0 \end{array}\right] \Rightarrow \text{rref} \Rightarrow \vec{I} = \begin{bmatrix} -.2449 \\ -.1114 \\ -.1166 \end{bmatrix}$$

2. The vectors $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ -2 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ -1 \\ 3 \end{bmatrix} \right\}$ a portion of (or all of) $\mathbb{R}^4$. Determine how many dimensions are in the span of the vectors (i.e. how many linearly independent vectors are there)? (5 points)

$$A = \begin{bmatrix} 1 & 2 & 4 & 3 \\ 2 & 1 & 2 & 3 \\ 0 & -1 & -2 & -1 \\ 2 & 1 & 2 & 3 \end{bmatrix} \Rightarrow \text{rref} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

2 pivots $\Rightarrow$ only spans a 2D subspace of $\mathbb{R}^4$
not all of $\mathbb{R}^4$
2 vectors are independent

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3. The matrix $A = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$ represents a rotation matrix. Find the angle of rotation. Give your answer in radians. If the result is not a standard angle, round to 4 decimal places. (5 points)

$\cos\theta = 1/\sqrt{5}$
$\sin\theta = 2/\sqrt{5}$

$\cos^{-1}(1/\sqrt{5}) = 1.107148\ldots$ radians

or $63.435°$

4. Determine if the linear transformation represented by the matrix $A = \begin{bmatrix} 4 & 2 & 1 \\ -2 & 1 & 3 \\ 3 & 0 & 1 \end{bmatrix}$ is one-to-one, onto, both or neither. (5 points)

rref $\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

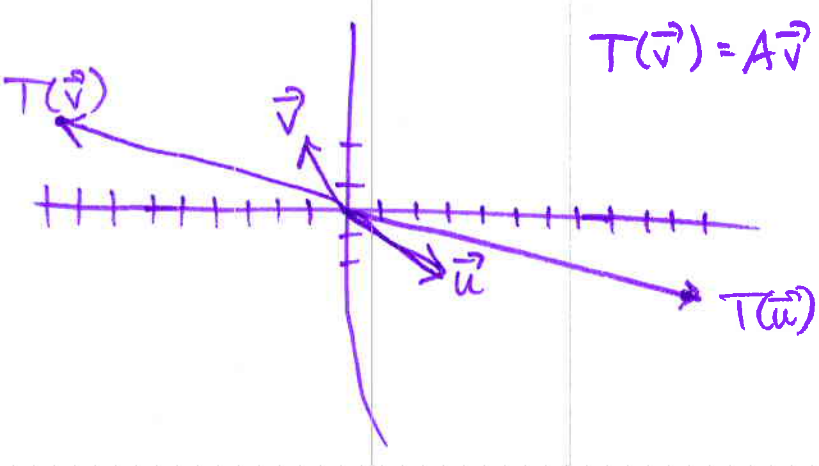
it is both onto & one-to-one

One-to-one since it has a pivot in every column.
onto since it has a pivot in every row.

5. The matrix $A = \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix}$ is a shear transformation. Plot the vectors $\vec{u} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ before the shear transformation and after applying it. You can use the same graph if all the vectors are properly labeled. (7 points)

$T(\vec{u}) = A\vec{u} = \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 3+8 \\ 0-2 \end{bmatrix} = \begin{bmatrix} 11 \\ -2 \end{bmatrix}$

$T(\vec{v}) = A\vec{v} = \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1-8 \\ 0+2 \end{bmatrix} = \begin{bmatrix} -9 \\ 2 \end{bmatrix}$



6. Describe in your own words how Nul A and Col A are related to each other (or contrast with each other). (4 points)

answers may vary:
Nul A is a subspace of the domain and is determined by the # of free variables.
Col A is a subspace of the ~~co~~ codomain and is determined by # of pivots. Their dimensions are related through $\dim \text{Col} A + \dim \text{Nul} A = n$ # of columns of A

7. Determine if the following sets are linearly independent or dependent. If the sets are dependent, find a basis for the subspace spanned by the vectors. Is the set a basis for the entire vector space? ($\mathbb{R}^5$ or $\mathbb{R}^3$ or P_3 respectively) (4 points each)

a. $\left\{\begin{bmatrix}0\\-1\\1\\2\\5\end{bmatrix},\begin{bmatrix}2\\3\\-2\\0\\6\end{bmatrix},\begin{bmatrix}-1\\1\\0\\1\\-3\end{bmatrix},\begin{bmatrix}4\\-1\\-3\\2\\0\end{bmatrix},\begin{bmatrix}3\\4\\-2\\0\\1\end{bmatrix}\right\}$ rref $\Rightarrow \begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{bmatrix}$

they are independent & span $\mathbb{R}^5$
So they are a basis for $\mathbb{R}^5$

b. $\left\{\begin{bmatrix}1\\-1\\-2\end{bmatrix},\begin{bmatrix}0\\0\\0\end{bmatrix},\begin{bmatrix}-2\\2\\-5\end{bmatrix}\right\}$ not independent because of $\vec{0}$
$\therefore$ not a basis for $\mathbb{R}^3$

basis for subspace is $\left\{\begin{bmatrix}1\\-1\\-2\end{bmatrix},\begin{bmatrix}-2\\2\\-5\end{bmatrix}\right\}$

c. $\{4-t^2, 6t+t^2-t^3, t^3+1\}$ P_3

$\begin{bmatrix}4\\0\\-1\\0\end{bmatrix}\begin{bmatrix}0\\6\\1\\-1\end{bmatrix}\begin{bmatrix}1\\0\\0\\1\end{bmatrix}$

does not span P_3
(since P_3 is 4 dimensional $\underbrace{a_0+a_1t+a_2t^2+a_3t^3}_{\text{4 constants}}$)

$\Rightarrow \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\\0&0&0\end{bmatrix}$ they are independent
they are a basis for the subspace (only)

8. Given the matrix $A = \begin{bmatrix} 1 & 2 & -3 & 4 & 0 & 1 \\ 0 & 3 & 4 & -1 & 1 & 1 \\ 2 & 3 & -2 & 1 & -2 & 0 \\ -1 & 0 & 1 & -1 & 1 & 0 \end{bmatrix}$, find the following:

a. Find an explicit description of Nul A. (6 points)

rref $\Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & -5/44 & -7/44 \\ 0 & 1 & 0 & 0 & 3/22 & 3/22 \\ 0 & 0 & 1 & 0 & 1/4 & 1/4 \\ 0 & 0 & 0 & 1 & 9/22 & 9/22 \end{bmatrix}$ free

$x_1 = 5/44\, x_5 + 7/44\, x_6$
$x_2 = -3/22\, x_5 - 3/22\, x_6$
$x_3 = -1/4\, x_5 - 1/4\, x_6$
$x_4 = -9/22\, x_5 - 9/22\, x_6$
$x_5 = x_5$
$x_6 = x_6$

$\vec{x} = \begin{bmatrix} 5/44 \\ -3/22 \\ -1/4 \\ -9/22 \\ 1 \\ 0 \end{bmatrix} x_5 + \begin{bmatrix} 7/44 \\ -3/22 \\ -1/4 \\ -9/22 \\ 0 \\ 1 \end{bmatrix} x_6$

or span $\left\{ \begin{bmatrix} 51 \\ -6 \\ -11 \\ -18 \\ 44 \\ 0 \end{bmatrix}, \begin{bmatrix} 7 \\ -6 \\ -11 \\ -18 \\ 0 \\ 44 \end{bmatrix} \right\}$

b. Find a basis for Col A. (3 points)

Col A = Span $\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 4 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ 1 \\ -1 \end{bmatrix} \right\}$

9. For each of the following questions, provide a short explanation with theoretical justifications. (4 points each)

a. What is a diagonal matrix? Define it and give an example.

a diagonal matrix are only entries on the main diagonal where both subscripts are the same a_{11}, a_{22}, etc and all other entries are zero.

e.g. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

b. Explain why R^3 is **NOT** a subspace of R^4.

vectors in $\mathbb{R}^3$ are determined by vectors w/ 3 components $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ whereas vectors in $\mathbb{R}^4$ have 4 components

$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ we can't just add the missing entry. They are not like things.