Stat 2470, Exam #2, Fall 2014 Name KEY

Instructions: Show all work. Use exact answers or appropriate rounding conventions. If you use your calculator, you can show work by saying which calculator commands you used.

1. Consider the discrete joint probability distribution shown below.

		y		
	$p(x,y)$	0	1	2
	0	.08	.04	.02
	1	.06	.10	.06
x	2	.05	.12	.08
	3	.04	.07	.10
	4	.01	.05	.12

a. Find $P(X = 2, Y = 1)$. (3 points)

.12

b. Find p_X and p_Y. (8 points)

p_X:

x	0	1	2	3	4
$p_X(x)$	.14	.22	.25	.21	.18

p_Y:

y	0	1	2
$p_Y(y)$	.24	.38	.38

c. Use your result in (b) to find $E(X)$ and $E(Y)$. (8 points)

$E(X) = 0(.14) + 1(.22) + 2(.25) + 3(.21) + 4(.18) = 2.07$

$E(Y) = 0(.24) + 1(.38) + 2(.38) = 1.14$

d. What is $E(XY)$? (6 points)

$E(XY) = 0(.08+.06+.05+.04+.01+.04+.02) + 1(.10) + 2(.02+.12) + 3(.07)$
$+ 4(.05+.08) + 6(.10) + 8(.12) = 2.67$

e. Use the results in (c) and (d) to determine whether X and Y are independent. (3 points)

$Cov(X,Y) = 2.67 - 2.07 * 1.14 = .3102$

Since the covariance is non-zero, they are not independent.

2. Suppose that $f(x,y) = \begin{cases} kx^4y^5, & 0 \le x \le 2, 0 \le y \le 2-x \\ 0, & elsewhere \end{cases}$.

a. Find the value of k that makes this a valid joint probability distribution. (7 points)

$$\int_0^2 \int_0^{2-x} kx^4y^5\,dy\,dx = \int_0^2 \frac{kx^4}{6} y^6 \Big|_0^{2-x} dx = k\int_0^2 \frac{x^4(2-x)^6}{6}\,dx$$

$$k \cdot \frac{512}{3465} = 1 \Rightarrow k = \frac{3465}{512}$$

b. Use the result in (a) to compute $E(X), E(Y), E(XY)$. (12 points)

$$E(X) = \int_0^2 \int_0^{2-x} \frac{3465}{512} x^5 y^5\,dy\,dx = \int_0^2 \frac{3465}{512 \cdot 6} x^5 (2-x)^6\,dx = \frac{5}{6}$$

$$E(Y) = \int_0^2 \int_0^{2-x} \frac{3465}{512} x^4 y^6\,dy\,dx = \int_0^2 \frac{3465}{512 \cdot 7} x^4 (2-x)^7\,dx = 1$$

$$E(XY) = \int_0^2 \int_0^{2-x} \frac{3465}{512} x^5 y^6\,dy\,dx = \int_0^2 \frac{3465}{512 \cdot 7} x^5 (2-x)^7\,dx = \frac{10}{13}$$

c. Use the results in (b) to find $Cov(X,Y)$. (3 points)

$$E(XY) - E(X)E(Y) = \frac{10}{13} - \frac{5}{6}(1) = -\frac{5}{78}$$

3. What is the rule of thumb for applying the Central Limit Theorem? Describe a circumstance when we might need to change the rule of thumb or particular distributions. (4 points)

generally we can apply it when $n \geq 30$

if the distribution is very skewed, we may need n to be much larger to obtain a normal sampling distribution

4. What is the condition we need to satisfy for an unbiased estimator? (4 points)

$$E(\hat{\theta}) = \theta$$

5. When we make a confidence statement, in reference, for instance, to a 95% confidence interval, what do we mean by this 95%? (4 points)

We mean that we are 95% certain that the true value of the parameter is inside the given interval.

Or, we can say that if we take samples of the same size we expect that $\frac{95}{100}$ we will obtain a result within the confidence interval.

6. Let X denote the proportion of allotted time that a randomly selected student spends working on a certain aptitude test, with a probability distribution function given by $f(x;\theta) = (\theta+1)x^{\theta}, 0 \le x \le 1,\ \theta > -1$. Suppose that 10 students have the following times: $\{0.92, 0.79, 0.90, 0.65, 0.86, 0.47, 0.73, 0.97, 0.77, 0.94\}$. Use this information to estimate the value of θ using the Maximum Likelihood Estimator function. (12 points)

$$L(\theta) = (\theta+1).92^{\theta} \cdot (\theta+1).79^{\theta} \cdot (\theta+1).90^{\theta} \cdot (\theta+1).65^{\theta} \cdot (\theta+1).86^{\theta} \cdot (\theta+1).47^{\theta} (\theta+1).73^{\theta} (\theta+1).97^{\theta} (\theta+1).77^{\theta} (\theta+1).94^{\theta}$$

$$= (\theta+1)^{10} (.0880806005)^{\theta}$$

$$\frac{dL}{d\theta} = 10(\theta+1)^{9}(.0880806005)^{\theta} + (\theta+1)^{10}(.0880806005)^{\theta} \cdot \ln(.0880806005)$$

$$= (\theta+1)^{9}(.0880806005)^{\theta}\left[10 + (\theta+1)\ln(.0880806005)\right] = 0$$

$$\frac{10 + \ln(.0880806005)}{-\ln(.0880806005)} = \frac{-\ln(.0880806005)\theta}{}$$

$$\theta \approx 3.116068237$$

7. The alternating current (AC) breakdown voltage for an insulating liquid indicates its dielectric strength. An article gave the following sample observations for a particular circuit under certain conditions specified in the article.

62	50	53	57	41	53	55	61	59	64
50	53	64	62	50	68	54	55	57	50
55	50	56	55	46	55	53	54	52	47
47	55	57	48	63	57	57	55	53	59
53	52	50	55	60	50	56	58		

a. Calculate a 95% confidence interval for the breakdown voltage. (6 points)

$s = 5.23067$

Z-Interval $(53.229, 56.188)$

T-Interval $(53.19, 56.227)$

b. What sample size would be required for the 95% confidence interval to have a width of 2 kV (so that μ is estimated to within 1 kV)? (4 points)

$$n = \left(2 z_{\alpha/2} \frac{\sigma}{w}\right)^2 \quad \text{or} \quad n = \left(2 \underbrace{t_{47, \alpha/2}}_{2.01174} \frac{s}{w}\right)^2$$

$$\text{or} \quad \left(2 \cdot 1.96 \frac{5.23067}{2}\right)^2$$

$$\approx 106$$

$$n = \left(2 \cdot 2.01174 \cdot \frac{5.23067}{2}\right)^2$$

$$= (10.5227)^2 = 110.728 \Rightarrow 111$$

c. What is the 95% prediction interval for the next measurement of breakdown voltage? (4 points)

$$\bar{x} \pm 2.01174 \cdot 5.23067 \sqrt{1 + \frac{1}{48}}$$

$$54.708 - 10.631795 = 44.076$$

$$54.708 + 10.631795 = 65.3398$$

$$(44.076, 65.3398)$$

8. If an estimate for the mean from a normally distributed population is taken from an sample of 150 measurements and found to be $\bar{x} = 31.4$ with a standard deviation $s = 2.8$. Calculate by hand the standard deviation for the sampling distribution. (4 points)

$$\frac{s}{\sqrt{n}} = \frac{2.8}{\sqrt{150}} = .2286$$

9. Find the values for the z and t values specified. (3 points each)

a. $z_{\alpha/2}$ for 99% confidence interval.

invNorm (.995) = 2.5758

b. $z_{\alpha/2}$ for $\alpha = 0.20$.

invNorm (.9) = 1.28

c. z_{α} for $\alpha = 0.05$ (one-sided)

invNorm (.05) = −1.64 or 1.64
left right

d. $t_{\alpha/2}$ for df=20, central area 0.95

$\alpha = .05$
$\alpha/2 = .025$

invT (.975, 20) = 2.0859

e. $t_{\alpha/2}$ for df=15, confidence level 99%

$a = .01$
$a/2 = .005$

invT(.995, 15) = 2.9467

f. t_{α} for df=5, for $\alpha = 0.05$ (one-sided)

invT(.05, 5) = −2.015 or 2.015
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