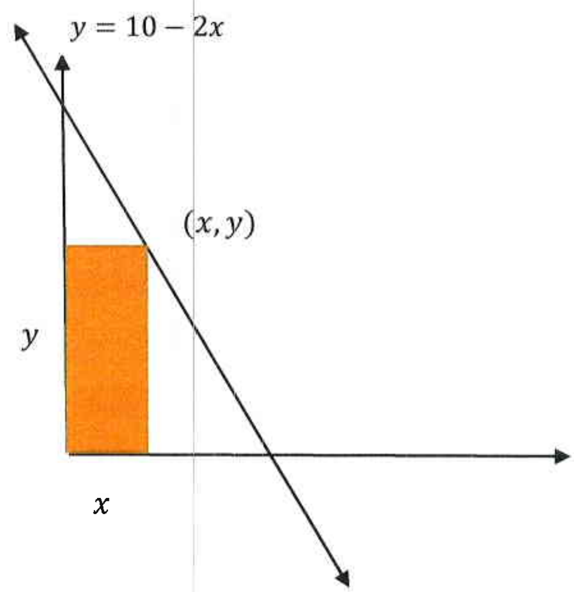
Math 1151, Final Exam, Fall 2014 Name KEY

Instructions: Show all work. Answers without work may only receive partial credit. If you are asked for an explanation, explain as completely as possible. Use exact answers unless specifically asked to round.

1. A rectangle is constructed with one side on the positive x-axis and one side on the positive y-axis, and the vertex on the line $y = 10 - 2x$. What dimensions maximize the area of the rectangle? What is the maximum area? (10 points)



$A = x(10-2x) = 10x - 2x^2$

$A' = 10 - 4x = 0$

$10 - 4x = 0$

$\frac{10}{4} = \frac{4x}{4}$

$x = \frac{5}{2}$

$y = 10 - 2\left(\frac{5}{2}\right) = 10 - 5 = 5$

$\frac{5}{2} \times 5 = \frac{25}{2}$

2. Make a sketch of the function $f(x) = \ln(2x)$ on the interval $[1, e]$. Determine whether the Mean Value Theorem applies on the given interval. If so, find the point(s) guaranteed to exist by the theorem. (8 points)

$\frac{\ln(2e) - \ln(2)}{e-1} = \frac{\cancel{\ln 2} + \ln e - \cancel{\ln 2}}{e-1} = \frac{1}{e-1}$

Continuous & differentiable on interval

$f'(x) = \frac{1}{2x} \cdot 2 = \frac{1}{x}$

$\frac{1}{c} = \frac{1}{e-1} \Rightarrow c = e-1$

3. Use L'Hôpital's Rule to find the indicated limits. Be sure to check that L'Hôpital's applies before proceeding. (6 points each)

a. $\lim\limits_{x\to 0}\frac{e^x-1}{x^2+3x}$ $\quad \frac{1-1}{0-0} = \frac{0}{0}$

$$\lim_{x\to 0}\frac{e^x}{2x+3} = \frac{1}{3} = \frac{1}{3}$$

b. $\lim\limits_{x\to 0} x\csc x$ $\quad = 0\cdot\infty \qquad \lim\limits_{x\to 0}\frac{x}{\sin x}$

$$\lim_{x\to 0}\frac{1}{\cos x} = 1$$

c. $\lim\limits_{x\to 0^+} x^{2x}$ $\quad 0^0 = L \qquad \lim\limits_{x\to 0^+} \ln x^{2x} = \lim\limits_{x\to 0^+} 2x\ln x = 0\cdot(-\infty)$

$$\lim_{x\to 0^+}\frac{2\ln x}{x^{-1}} = \lim_{x\to 0^+}\frac{2\cdot\frac{1}{x}}{-\frac{1}{x^2}}\cdot\frac{x^2}{x^2} = \lim_{x\to 0^+}\frac{2x}{-1} = 0 = \ln L$$

$$L = e^0 = 1$$

4. Find the antiderivatives for the following functions. (7 points each)

a. $\int 3x^{-2} - 6\sqrt[3]{x^2} + 9e^{-7x} + x\,dx$

$6x^{2/3}$

$$-3x^{-1} + -6\cdot\frac{3}{5}x^{5/3} + \frac{9}{-7}e^{-7x} + \frac{1}{2}x^2 + C$$

$$\frac{-3}{x} - \frac{18}{5}x^{5/3} - \frac{9}{7}e^{-7x} + \frac{1}{2}x^2 + C$$

b. $\int \frac{12t^8 - t}{t^2} dt$ $= \int 12t^6 - \frac{1}{t} \, dt = \frac{12}{7} t^7 - \ln t + C$

c. $\int \frac{6}{x^2+49} dx$

$\frac{6}{7} \arctan\left(\frac{x}{7}\right) + C$

$= \frac{6}{7} \tan^{-1}\left(\frac{x}{7}\right) + C$

d. $\int 4^{\sin x} \cos x \, dx$

$u = \sin x$

$du = \cos x \, dx$

$\int 4^u \, du = \frac{4^u}{\ln 4} + C = \frac{4^{\sin x}}{\ln 4} + C$

e. $\int \frac{8x+6}{2x^2+3x} dx$

$u = 2x^2 + 3x$

$du = (4x+3) \, dx$

$\int \frac{2(4x+3)}{2x^2+3x} dx = 2 \int \frac{du}{u} = 2 \ln|u| + C = 2 \ln|2x^2+3x| + C$

f. $\int \cot x \, dx$ [Hint: rewrite in terms sine and cosine functions.]

$\int \frac{\cos x}{\sin x} dx$

$u = \sin x$

$du = \cos x \, dx$

$\int \frac{1}{u} du = \ln|u| + C = \ln|\sin x| + C$

g. $\int \operatorname{sech}^2 x \tanh x \, dx$

$u = \tanh x$

$du = \operatorname{sech}^2 x \, dx$

$\int u \, du = \frac{1}{2}u^2 + C = \frac{1}{2}\tanh^2 x + C$

h. $\int \frac{\sinh \ln x}{x} dx$

$u = \ln x$

$du = \frac{1}{x} dx$

$\int \sinh u \, du = \cosh u + C = \cosh(\ln x) + C$

5. Use the limit definition of the definite integral to find the area under the curve $y = -3x + 12$ on the interval $[-1,2]$. Sketch the graph of the area. (10 points)

$\Delta x = \frac{2+1}{n} = \frac{3}{n}$ $\quad x_i = -1 + \frac{3i}{n}$

$f(x_i) = -3(-1 + \frac{3i}{n}) + 12 = 3 - \frac{9i}{n} + 12 = 15 - \frac{9i}{n}$

$\sum f(x_i)\Delta x = \sum (15 - \frac{9i}{n})(\frac{3}{n}) = \sum \left(\frac{45}{n} - \frac{27i}{n^2}\right) =$

$\sum \frac{45}{n} - \frac{27}{n^2}\sum i = \frac{45}{\cancel{n}} \cdot \cancel{n} - \frac{27}{n^{\cancel{2}}} \cdot \frac{\cancel{n}(n+1)}{2} =$

$45 - \frac{27\cancel{n}}{2\cancel{n}} - \frac{27}{2n} = \frac{45}{1} - \frac{27}{2} - \frac{27}{2n} = \frac{90}{2} - \frac{27}{2} - \frac{27}{2n} =$

$\lim_{n \to \infty} \frac{63}{2} - \frac{27}{2n} = \frac{63}{2}$

6. Use the Fundamental Theorem of Calculus and properties of definite integrals to evaluate the following. (5 points each)

a. $\int_0^1 x - \sqrt{x}\,dx$

$\int_0^1 x - x^{1/2}\,dx$

$\frac{1}{2}x^2 - \frac{2}{3}x^{3/2}\Big|_0^1 = \frac{1}{2}(1)^2 - \frac{2}{3}(1)^{3/2} - 0 = \frac{1}{2} - \frac{2}{3} = \frac{-1}{6}$

b. $\int_{-2}^{2} x^2 - 4\,dx$

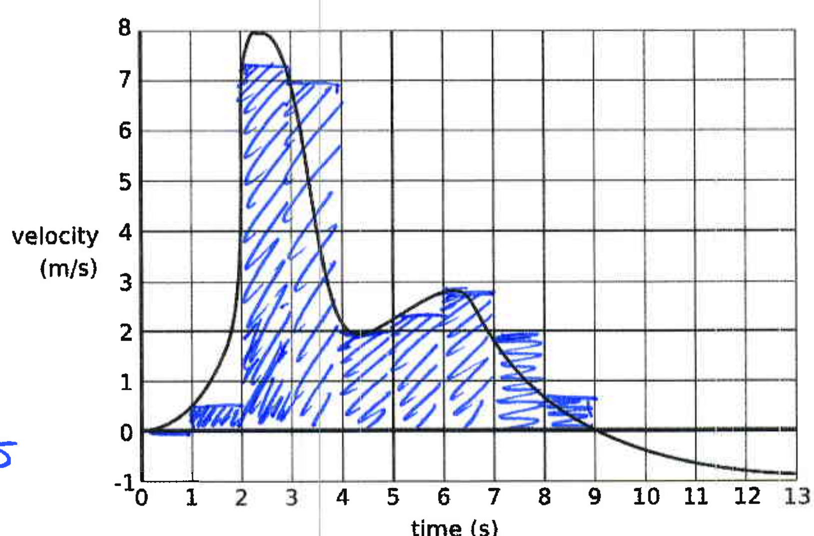
even $= 2\int_0^2 x^2 - 4\,dx = 2\left[\frac{1}{3}x^3 - 4x\Big|_0^2\right. =$

$2\left[\frac{8}{3} - 8 - 0\right] = 2\left(-\frac{16}{3}\right) = -\frac{32}{3}$

c. $\int_{-\pi}^{\pi} \sin(x)\,dx$

odd $= 0$

7. Use the graph of the velocity function of an object below to approximate the displacement of the function from time $t = 0$ to $t = 9$ (with nine rectangles). Then use the same graph to determine where any points of inflection are on the graph of the position function. (10 points)



inflection points at $\approx t = 2.5$
$t = 4.25$
$t = 6.3$

using left-hand rule

$0 + \frac{1}{2} + 7\frac{1}{4} + 7 + 2 + 2\frac{1}{4} + 2\frac{3}{4} + 2 + \frac{3}{4} = 24.5 = \frac{49}{2}$

(or similar result)

8. Use Simpson's Rule to approximate the area under the function $f(x) = \ln x$ on the interval $[1,3]$, using n=6. (10 points)

$\Delta x = \frac{2}{6} = \frac{1}{3}$ $x_0 = 1$, $x_1 = 4/3$, $x_2 = 5/3$, $x_3 = 2$,
$x_4 = \frac{7}{3}$, $x_5 = \frac{8}{3}$, $x_6 = 3$

$\frac{\cancel{2}1}{3 \cdot \cancel{6}_3}\left[\ln 1 + 4\ln 4/3 + 2\ln 5/3 + 4\ln(2) + 2\ln(7/3) + 4\ln(8/3) + \ln 3\right] =$

$\frac{1}{9}[11.66\ldots] = 1.2957$

9. Find the derivatives of the following functions. (5 points each)

a. $f(x) = \sec x + \tan x$

$\sec x \tan x + \sec^2 x$

b. $y = e^{-x} \sin x$

$-e^{-x} \sin x + e^{-x} \cos x$

c. $g(x) = (3x^2 + 7x)^{10}$

$10(3x^2+7x)^9(6x+7)$

d. $h(t) = e^{\tan t}$ $\quad e^{\tan t} \sec^2 t$

e. $q(t) = 5^{3t}$ $\quad 5^{3t} \cdot 3 \cdot \ln 5$

f. $y = \tan^{-1}(x^2)$

$$\frac{1}{1+(x^2)^2} \cdot 2x = \frac{2x}{1+x^4}$$

g. $v(x) = x^x$ $\qquad y = x^x \qquad \ln y = \ln x \cdot x$

$$\frac{1}{y} y' = \frac{1}{x} \cdot x + \ln x$$

$$y' = v'(x) = [1 + \ln x] x^x$$

h. $w(x) = \cosh x + \operatorname{sech} x^2$

$$\sinh x + \operatorname{sech} x^2 \cdot \tanh x^2 \cdot 2x$$

10. Find the equation of the tangent line to the graph of $f(x) = x^3 - 4x^2 + 2x - 1$ when $x = 2$. (6 points)

$y' = 3x^2 - 8x + 2 \qquad f'(2) = 3 \cdot 4 - 8 \cdot 2 + 2 = 12 - 16 + 2 = -2$

$f(2) = 8 - 4(4) + 2(\cdot 2) - 1 = 8 - 16 + 4 - 1 = -5 \qquad (2, -5)$

$$y + 5 = -2(x - 2)$$
$$y + 5 = -2x + 4$$
$$y = -2x - 1$$

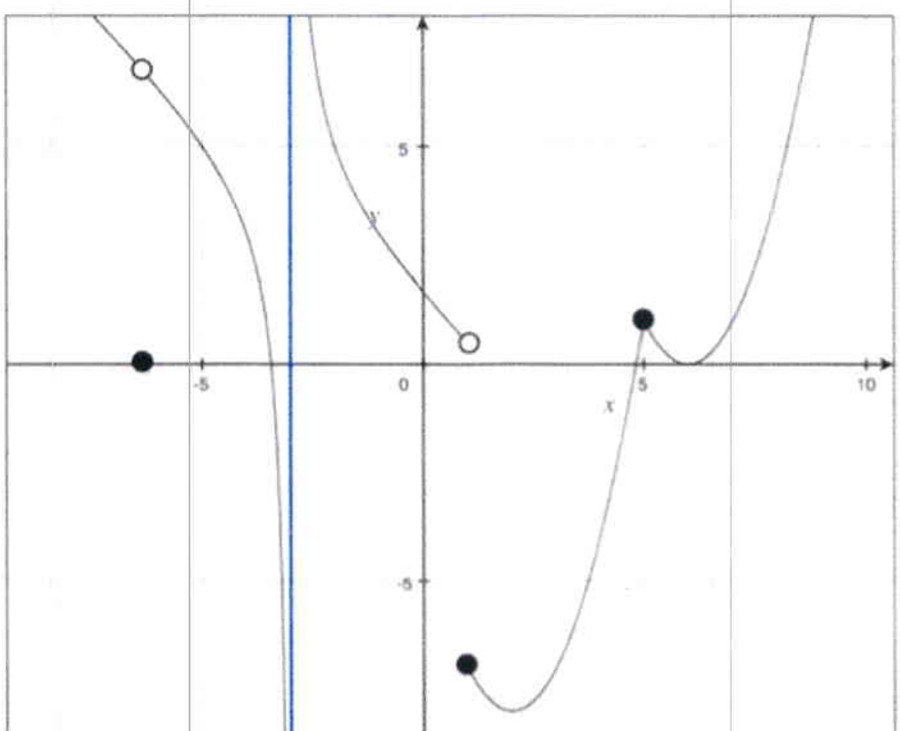
11. Use the limit definition of the derivative to find the equation of the derivative of the following function $f(x) = x - x^2$. (8 points)

$$\lim_{\Delta x \to 0} \frac{x+\Delta x - (x+\Delta x)^2 - x + x^2}{\Delta x} = \lim_{\Delta x \to 0} \frac{\cancel{x}+\Delta x - \cancel{x^2} - 2x\Delta x - \Delta x^2 - \cancel{x} + \cancel{x^2}}{\Delta x}$$

$$\lim_{\Delta x \to 0} \frac{\Delta x - 2x\Delta x - \Delta x^2}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta x(1-2x-\Delta x)}{\Delta x} = \lim_{\Delta x \to 0} 1-2x-\Delta x =$$

$$1-2x$$

12. Shown below is the graph of $f(x)$. Find the limit at the indicated values. (2 points each)



a. $\lim_{x \to 5} f(x) = 1$

c. $\lim_{x \to 6} f(x) = 0$

b. $\lim_{x \to -3^-} f(x) = -\infty$

d. $\lim_{x \to 1} f(x) = \text{DNE}$

13. Find the limit algebraically of the following. (5 points each)

a. $\lim_{h\to 0}\frac{\sqrt{9+h}-3}{h} \cdot \frac{\sqrt{9+h}+3}{\sqrt{9+h}+3} = \lim_{h\to 0}\frac{\cancel{9}+h-\cancel{9}}{h(\sqrt{9+h}+3)} = \lim_{h\to 0}\frac{\cancel{h}}{\cancel{h}(\sqrt{9+h}+3)} =$

$\lim_{h\to 0}\frac{1}{\sqrt{9+h}+3} = \frac{1}{\sqrt{9}+3} = \frac{1}{3+3} = \frac{1}{6}$

b. $\lim_{x\to 0}\frac{\tan(5x)}{3x} = \lim_{x\to 0}\frac{\sin 5x}{\cos 5x}\cdot\frac{1}{3x}\cdot\frac{5}{5} =$

$\lim_{x\to 0}\frac{\sin 5x}{5x}\cdot\frac{5}{3\cos 5x} = \lim_{x\to 0}\frac{\sin 5x}{5x}\cdot\lim_{x\to 0}\frac{5}{3\cos 5x} =$

$1\cdot\frac{5}{3} = \frac{5}{3}$

14. Is the function $f(x) = \begin{cases} 2x^2 - x, & x < 1 \\ -\frac{2}{3}x + \frac{5}{3}, & x \geq 1 \end{cases}$ continuous on its domain? Check for continuity at $x = 1$. (8 points)

each piece is continuous

$\lim_{x\to 1^-} 2x^2 - x = 2(1)^2 - 1 = 1$

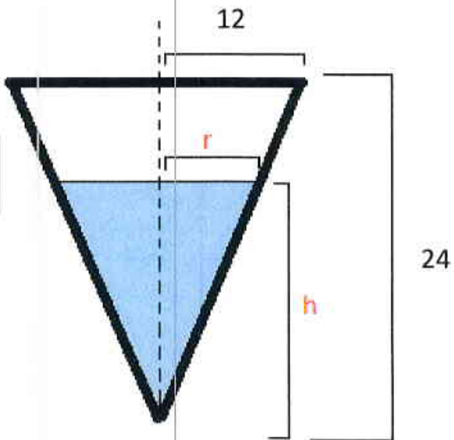
$\lim_{x\to 1^+} -\frac{2}{3}x + \frac{5}{3} = -\frac{2}{3}(1) + \frac{5}{3} = 1$

$\lim_{x\to 1} f(x) = 1$

$f(1) = 1$

it is continuous everywhere

15. An inverted conical water tank with a height of 24 ft and a radius of 12 ft is drained through a hole in the vertex at the rate of 2 ft^3/s. What is the rate of change of the water depth when the water depth is 6 feet? [Hint: use similar triangles. The volume of a cone is $V = \frac{1}{3}\pi r^2 h$.] (10 points)



$\frac{r}{h} = \frac{12}{24} = \frac{1}{2}$ $\quad 2r = h$

$\frac{h}{2} = r$

$\frac{dv}{dt} = 2$

$h = 6$

$V = \frac{1}{3}\pi r^2 h$

$= \frac{1}{3}\pi \left(\frac{h}{2}\right)^2 h = \frac{1}{12}\pi h^3$

$\frac{dv}{dt} = \frac{3}{12}\pi h^2 = \frac{1}{4}\pi h^2 \frac{dh}{dt}$

$2 = \frac{1}{4}\pi (6)^2 \frac{dh}{dt}$

$\frac{8}{36\pi} = \frac{dh}{dt} = \frac{2}{9\pi}$

16. Find the absolute maximum and minimum of the function $f(x) = 2x^6 - 15x^4 + 24x^2$ on $[-2,7]$. (8 points)

$f'(x) = 12x^5 - 60x^3 + 48x$

$12x(x^4 - 5x^2 + 4)$

$12x(x^2-1)(x^2-4)$

$12x(x-1)(x+1)(x+2)(x-2)$

$x = 0, 1, -1, 2, -2$

$f(-2) = -16$
$f(-1) = -11$
$f(0) = 0$
$f(1) = 11$
$f(2) = -16$
$f(7) = 200,459$

abs mins @ ± 2 $\quad y = -16$

abs max @ 7 $\quad y = 200,459$

17. Find the critical points and inflection points of the function $y = x^2e^{-x}$. Develop sign charts for both the first and second derivatives and use that information to determine whether each critical point is a maximum or a minimum. Sketch the function. (10 points)

$$f'(x) = 2xe^{-x} - x^2e^{-x} = xe^{-x}(2-x) = e^{-x}(2x-x^2)$$

$$x=0,\ x=2$$

$$f''(x) = -e^{-x}(2x-x^2) + e^{-x}(2-2x)$$

$$= e^{-x}(-2x+x^2+2-2x) = e^{-x}(x^2-4x+2)$$

$$\frac{4 \pm \sqrt{16-8}}{2} = \frac{4\pm\sqrt{8}}{2} = \frac{4 \pm 2\sqrt{2}}{2} = 2 \pm \sqrt{2}$$

$$\approx .6,\ 3.4$$

f': $-$ | 0 | $+$ | 2 | $-$

f'': $+$ | .6 ($2-\sqrt{2}$) | $-$ | 3.4 ($2+\sqrt{2}$) | $+$