HOMOGENEOUS SOLUTIONS AND NULLSPACES

There are a number of applications in linear algebra that depend on the solution to systems of equations that are homogeneous. For a system to be homogeneous, we need to be able to write all the equations in the system in the form $a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0$, or in matrix form, the system must be written as $A\vec{x} = \vec{0}$. Systems of this sort always have a solution where every $x_i = 0$ will always solve the equation since there are no constants (this is called the trivial solution of a homogeneous system since every system has this solution and it's pretty boring), and the system is always consistent. However, sometimes there will be solutions where the all the variables are not all zero, and so before we can discuss these more interesting solutions of homogeneous systems (called non-trivial solutions), we will first have to discuss how to solve dependent systems, systems with infinite numbers of solutions.

Dependent systems of equations arise when there is not enough information to solve for every variable exactly. We first see this when we start out with fewer equations than we have variables, as in the system below.

$$3x_1 + 2x_2 - x_3 = 7$$
$$2x_1 - 4x_2 - 5x_3 = 8$$

Since we don't have enough variables to get a single solution to the system, we are going to have one variable that we can choose freely, i.e. it can be any real number. Once selected, the rest of the system will be determined. To express the solution, then, we are going to need to write the system in parametric form. This will express the relationships among the variables in a compact form.

Example 1. Solve the system above and write the solution in parametric form.

To solve the system, we are first going to write it as an augmented matrix, and then row reduce the system.

$$3x_1 + 2x_2 - x_3 = 7$$
$$2x_1 - 4x_2 - 5x_3 = 8$$

$$\begin{bmatrix} 3 & 2 & -1 & | & 7 \\ 2 & -4 & -5 & | & 8 \end{bmatrix}$$

Which in turn reduces to $\begin{bmatrix} 1 & 0 & -\frac{7}{8} & | & \frac{11}{4} \\ 0 & 1 & \frac{13}{16} & | & -\frac{5}{8} \end{bmatrix}$ (in *reduced* echelon form; it is necessary to go "all the way" for all parametric solutions). From this point, write the system as a set of equations again.

$$x_1 - \frac{7}{8}x_3 = \frac{11}{4}$$
$$x_2 + \frac{13}{16}x_3 = -\frac{5}{8}$$

Solve the first equation for x_1, and the second equation for x_2. Since there is no third equation for x_3, this is our free variable, so let the third equation just say $x_3 = x_3$.

$$x_1 = \frac{7}{8}x_3 + \frac{11}{4}$$
$$x_2 = -\frac{13}{16}x_3 - \frac{5}{8}$$
$$x_3 = x_3$$

Recall that the solution to our equation must have a solution for all of the variables in the system. So even though we have only two equations, the vector solution must have three components for this example since there are three variables. We let $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \vec{x}$ on the left side. On the right side, we pull out the coefficients for x_3, and then collect the constants in a second vector as shown below.

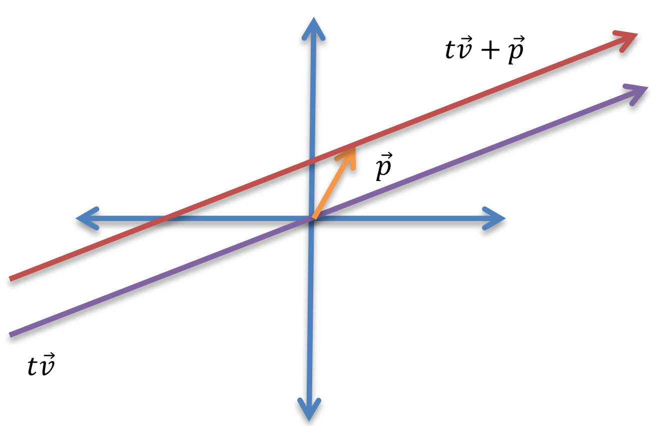
$$\vec{x} = \begin{bmatrix} \frac{7}{8} \\ -\frac{13}{16} \\ 1 \end{bmatrix} x_3 + \begin{bmatrix} \frac{11}{4} \\ -\frac{5}{8} \\ 0 \end{bmatrix}$$

This expresses all the possible solutions of the system. If we $x_3 = -2$, for instance, we get a solution for $\vec{x} = \begin{bmatrix} \frac{7}{8} \\ -\frac{13}{16} \\ 1 \end{bmatrix}(-2) + \begin{bmatrix} \frac{11}{4} \\ -\frac{5}{8} \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{7}{4} \\ \frac{13}{8} \\ -2 \end{bmatrix} + \begin{bmatrix} \frac{11}{4} \\ -\frac{5}{8} \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$.

Because the vector multiplying x_3 is being scaled by a real number, it is often convenient to choose a parameter capable of eliminating the fractions in the vector. Here, we can let $16t = x_3$, and then the solution will become

$$\vec{x} = \begin{bmatrix} 14 \\ -13 \\ 16 \end{bmatrix} t + \begin{bmatrix} \frac{11}{4} \\ -\frac{5}{8} \\ 0 \end{bmatrix}$$

Note that we can't scale the constant vector since this vector isn't being scaled by a real number.



This parametric version of the solution is written as $\vec{x} = t\vec{v} + \vec{p}$, where $\vec{v}$ is the scaled vector, and $\vec{p}$ is the constant vector. We think of the solution like this with one free variable as a line in space $t\vec{v}$ shifted off the origin by $\vec{p}$.

In the case of the homogenous problem, the $t\vec{v}$ portion of the solution (or several of these depending on the number of free variables) will be present, but the $\vec{p}$ component will be zero, since we will have started out with all our constants already equal to zero, we want our final solution to go through the origin (the equivalent of having the trivial $\vec{0}$ solution as one of the solutions to the system).

Example 2. Solve the system below and write any non-trivial solution in parametric form.

$$\begin{aligned} x_1 + 2x_2 - x_3 + 5x_4 &= 0 \\ -2x_1 - 4x_2 + 2x_3 - 10x_4 &= 0 \\ 2x_1 + 2x_2 + x_3 - 3x_4 &= 0 \\ 3x_1 + 4x_2 \qquad + 2x_4 &= 0 \end{aligned}$$

We can write the system as an augmented matrix as

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 5 & 0 \\ -2 & -4 & 2 & -10 & 0 \\ 2 & 2 & 1 & -3 & 0 \\ 3 & 4 & 0 & 2 & 0 \end{array}\right]$$

However, because row operations are not going to affect the column of zeros at the end, we usually just work with the coefficient portion of the matrix shown below, with the augment understood when we go back to the system of equations at the end.

$$\begin{bmatrix} 1 & 2 & -1 & 5 \\ -2 & -4 & 2 & -10 \\ 2 & 2 & 1 & -3 \\ 3 & 4 & 0 & 2 \end{bmatrix}$$

We can reduce this matrix to the following:

$$\begin{bmatrix} 1 & 0 & 2 & -8 \\ 0 & 1 & -\frac{3}{2} & \frac{13}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This system has two free variables. As with Example 1, we start by rewriting the reduced system as a pair of equations, and declare our free variables

$$\begin{aligned} x_1 + 2x_3 - 8x_4 &= 0 \\ x_2 - \frac{3}{2}x_3 + \frac{13}{2}x_4 &= 0 \\ x_3 &= x_3\ (free) \\ x_4 &= x_4\ (free) \end{aligned}$$

We move the free variables in the first two equations to the other side and then pull out the coffecients of each variable.

$$x_1 = -2x_3 + 8x_4$$
$$x_2 = \frac{3}{2}x_3 - \frac{13}{2}x_4$$
$$x_3 = x_3$$
$$x_4 = x_4$$

$$\vec{x} = \begin{bmatrix} -2 \\ \frac{3}{2} \\ 1 \\ 0 \end{bmatrix} x_3 + \begin{bmatrix} 8 \\ -\frac{13}{2} \\ 0 \\ 1 \end{bmatrix} x_4$$

Or, if we let $x_3 = 2t \; and \; x_4 = 2s$, this can also be written as:

$$\vec{x} = \begin{bmatrix} -4 \\ 3 \\ 2 \\ 0 \end{bmatrix} t + \begin{bmatrix} 15 \\ -13 \\ 0 \\ 2 \end{bmatrix} s$$

You'll notice that because there are two free variables, this system is built on a linear combination of two different vectors, both of which can be scaled freely. The other thing to notice is that the coefficient matrix was 4x4 (in general mxn), and the solution vectors must match the n dimension (the number of columns) of the coefficient matrix. This is necessary since the solution must satisfy the equation $A\vec{x} = \vec{0}$, and in order for that multiplication to be defined, the solution must be nx1. With a square matrix, you may accidentally get the correct size since it's the same in both directions, but in Example 1 we had a 2x3 coefficient matrix, and our solutions were all 3x1 vectors. Until this becomes automatic, it's a good idea to check the dimensions of the solution with the coefficient matrix using dimensional analysis to make sure the multiplication is defined.

When we move into the realm of linear transformations and a little bit more abstraction, we encounter solutions of homogeneous solutions again when we encounter the nullspace of a matrix.

The **nullspace** of a linear transformation is the set of vectors that map onto the zero vector $\vec{0}$ in the codomain. The nullspace is also called the **kernel** of a linear transformation. Because this is essentially the solution to the homogeneous system, we will solve for the nullspace in the same way we did in Example 2. However, in these problems, we generally do not have a system to start with, just the matrix of the transformation. Another difference is that we will list the vectors in a set for the solution, rather than keep it in parametric form.

Example 3. Find the basis of the nullspace for the linear transformation shown below.

$$A = \begin{bmatrix} 1 & 2 & 4 & 0 & 1 \\ -1 & 3 & 1 & 1 & 0 \\ 3 & -1 & 5 & -2 & 1 \end{bmatrix}$$

As before, we must row-reduce the matrix to reduced echelon form.

$$\begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

As we did in Example 2, we write the reduced matrix as a system of equations and declare the free variables.

$$x_1 + 2x_3 + x_5 = 0$$
$$x_2 + x_3 = 0$$
$$x_4 + x_5 = 0$$
$$x_3 = x_3\ (free)$$
$$x_5 = x_5\ (free)$$

As before, move all the free variables to the other side. We also have to order the equations so that the left side of the equations have the variables appear in order.

$$x_1 = -2x_3 - x_5$$
$$x_2 = -x_3$$
$$x_3 = x_3$$
$$x_4 = -x_5$$
$$x_5 = x_5$$

And then we pull out the coefficients of $x_3\ and\ x_5$.

$$\vec{x} = \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} x_3 + \begin{bmatrix} -1 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} x_5$$

Therefore the basis of the nullspace is $NulA = \left\{ \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}$. Any linear combination of these vectors will map unto the zero vector if multiplied by the original A matrix. Again, you'll note that the original matrix was 3x5, and our solution vectors are 5x1.

Solving for the solution to a homogeneous system or the nullspace comes up in a number of applications in linear algebra, such as solving for equilibrium vectors (in Markov chains) or solving for eigenvectors.

In a Markov Chain, if we want to find the equilibrium vector, we want to solve for a vector that, if multiplied by the Markov chain matrix, reproduces the original vector, or in equation form, we want to find $\vec{q}$ such that $P\vec{q} = \vec{q}$. We can solve this by converting it to a homogeneous system as shown below.

$$P\vec{q} - \vec{q} = 0$$
$$P\vec{q} - I\vec{q} = 0$$
$$(P - I)\vec{q} = 0$$

We can find $\vec{q}$ then by solving for the nullspace of the P-I matrix, where I is a suitable nxn identity matrix.

Similarly, to solve for the eigenvector of a matrix, with a given eigenvalue λ, we want to solve the equation $A\vec{v} = \lambda\vec{v}$. As with the Markov Chain problem, we will collect terms on one side and solve a homogeneous system.

$$A\vec{v} - \lambda\vec{v} = 0$$
$$A\vec{v} - \lambda I\vec{v} = 0$$
$$(A - \lambda I)\vec{q} = 0$$

Then we solve for the nullspace of A-λI, where again, I is a suitable nxn identity matrix.

The details of Markov Chains and Eigenvectors are dealt with in another handout.

Practice Problems.

1. Solve the system of equations and write the solution in parametric form.
   a. $\begin{cases} x_1 + 2x_2 - 3x_3 = 5 \\ 2x_1 + x_2 - 3x_3 = 13 \\ -x_1 + x_2 \qquad = -8 \end{cases}$
   b. $\begin{cases} 2x_1 + 2x_2 + 4x_3 = 0 \\ -4x_1 - 4x_2 - 8x_3 = 0 \\ \quad -3x_2 - 3x_3 = 0 \end{cases}$
   c. $\begin{cases} 5x_1 - 3x_2 + 2x_3 = 0 \\ -3x_1 - 4x_2 + 2x_3 = 0 \end{cases}$
   d. $\begin{cases} x_1 + 2x_2 - 3x_3 + x_4 - x_5 = 1 \\ 2x_1 - 2x_2 + 5x_3 - 2x_4 - x_5 = 6 \\ x_1 + 4x_2 - 4x_3 + x_4 + 2x_5 = 3 \\ -3x_1 + 5x_2 - 9x_3 - 8x_4 + 5x_5 = -7 \end{cases}$
2. Find the basis of the nullspace of each linear transformation.
   a. $A = \begin{bmatrix} 1 & -3 & -8 & 5 \\ 0 & 1 & 2 & -4 \end{bmatrix}$
   b. $A = \begin{bmatrix} 1 & -4 & -2 & 0 & 3 & -5 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$
   c. $A = \begin{bmatrix} 10 & -8 & -2 & -2 \\ 0 & 2 & 2 & -2 \\ 1 & -1 & 6 & 0 \\ 1 & 1 & 0 & -2 \end{bmatrix}$
   d. $A = \begin{bmatrix} 5 & -2 & 3 \\ -1 & 0 & -1 \\ 0 & -2 & -2 \\ 5 & 7 & 2 \end{bmatrix}$
3. Find the kernel of each linear transformation.
   a. $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 - x_3 \\ -x_1 + 3x_2 + x_3 \end{bmatrix}$
   b. $T\left(\begin{bmatrix} r \\ s \\ t \end{bmatrix}\right) = \begin{bmatrix} 2s + t \\ r - s + 2t \\ 3r + s \\ 2r - s - t \end{bmatrix}$

c. $A = \begin{bmatrix} 1 & 2 & -4 & 3 & -2 & 6 & 0 \\ 0 & 0 & 0 & 1 & 0 & -3 & 7 \\ 0 & 0 & 0 & 0 & 1 & 4 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

d. $A = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$