Math 2568 Markov Chains

1a. $P = \begin{bmatrix} .7 & .4 \\ .3 & .6 \end{bmatrix}$ $\quad P - I = \begin{bmatrix} -.3 & .4 \\ .3 & -.4 \end{bmatrix}$ $\quad .3x_1 - .4x_2 = 0$

$\Rightarrow \frac{.3x_1}{.3} = \frac{.4x_2}{.3}$

$x_1 = 4/3\, x_2 \Rightarrow \vec{x} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$ $\quad x_1 + x_2 = 7$

$\vec{q} = \begin{bmatrix} 4/7 \\ 3/7 \end{bmatrix}$ there is just one vector. The variables interact w/ each other.

b. $P = \begin{bmatrix} .5 & .04 \\ .5 & .96 \end{bmatrix}$ $\quad P - I = \begin{bmatrix} -.5 & .04 \\ .5 & -.04 \end{bmatrix}$ $\quad .5x_1 - .04x_2 = 0$

$x_1 = \frac{.04}{.5} x_2$

$x_1 = .08x_2 = \frac{2}{25}$ $\quad \vec{x} = \begin{bmatrix} 2 \\ 25 \end{bmatrix}$ $\quad x_1 + x_2 = 27 \Rightarrow$

$\vec{q} = \begin{bmatrix} 2/27 \\ 25/27 \end{bmatrix}$ there is just one vector, as above.

c. $P = \begin{bmatrix} .99 & .08 \\ .01 & .92 \end{bmatrix}$ $\quad P - I = \begin{bmatrix} -.01 & .08 \\ .01 & -.08 \end{bmatrix} \Rightarrow$ $\quad .01x_1 - .08x_2 = 0$

$x_1 = \frac{.08}{.01} x_2$

$x_1 = 8x_2$ $\quad \vec{x} = \begin{bmatrix} 8 \\ 1 \end{bmatrix}$ $\quad x_1 + x_2 = 9 \Rightarrow \vec{q} = \begin{bmatrix} 8/9 \\ 1/9 \end{bmatrix}$

one vector, as above.

d. $P = \begin{bmatrix} .4 & .3 & .1 \\ .3 & .5 & .2 \\ .3 & .2 & .7 \end{bmatrix}$ $\quad P - I = \begin{bmatrix} -.6 & .3 & .1 \\ .3 & -.5 & .2 \\ .3 & .2 & -.3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & -11/21 \\ 0 & 1 & -5/7 \\ 0 & 0 & 0 \end{bmatrix}$

$x_1 = 11/21\, x_3$

$x_2 = 5/7\, x_3$

$x_3 = 1\, x_3$

$\vec{x} = \begin{bmatrix} 11 \\ 15 \\ 21 \end{bmatrix}$ $\quad x_1 + x_2 + x_3 = 47$ $\quad \vec{q} = \begin{bmatrix} 11/47 \\ 15/47 \\ 21/47 \end{bmatrix}$

one vector, as above

1e. $P = \begin{bmatrix} .88 & .03 & .15 \\ .08 & .75 & .21 \\ .04 & .22 & .64 \end{bmatrix}$ $P - I = \begin{bmatrix} -.12 & .03 & .15 \\ .08 & -.25 & .21 \\ .04 & .22 & -.36 \end{bmatrix} \Rightarrow$

$\begin{bmatrix} 1 & 0 & -73/46 \\ 0 & 1 & -31/23 \\ 0 & 0 & 0 \end{bmatrix}$ $x_1 = 73/46\, x_3$ $\quad \vec{x} = \begin{bmatrix} 73 \\ 62 \\ 46 \end{bmatrix}$

$x_2 = 31/23\, x_3$

$x_3 = x_3$

$x_1 + x_2 + x_3 = 181$ $\quad \vec{q} = \begin{bmatrix} 73/181 \\ 62/181 \\ 46/181 \end{bmatrix}$

One vector, as above

f. $P = \begin{bmatrix} .9 & .002 & .12 \\ .099 & .95 & .004 \\ .001 & .048 & .876 \end{bmatrix}$ $P - I = \begin{bmatrix} -.1 & .002 & .12 \\ .099 & -.05 & .004 \\ .001 & .048 & -.124 \end{bmatrix} \Rightarrow$

$\begin{bmatrix} 1 & 0 & -3004/2401 \\ 0 & 1 & -6140/2401 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow$ $x_1 = \frac{3004}{2401} x_3$ $\quad \vec{x} = \begin{bmatrix} 3004 \\ 6140 \\ 2401 \end{bmatrix}$

$x_2 = \frac{6140}{2401} x_3$

$x_3 = x_3$

$x_1 + x_2 + x_3 = 11,545$

$\vec{q} = \begin{bmatrix} 3004/11,545 \\ 1228/2309 \\ 2401/11545 \end{bmatrix}$ one vector, above

g. $P = \begin{bmatrix} .8 & 0 & .5 \\ .1 & 1 & .1 \\ .1 & 0 & .4 \end{bmatrix}$ $P - I = \begin{bmatrix} -.2 & 0 & .5 \\ .1 & 0 & .1 \\ .1 & 0 & -.6 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$x_1 = 0$
$x_2 = x_2 \Rightarrow \vec{q} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$
$x_3 = 0$

One vector

all the objects end up in box B since once its there it can't leave.

h. $P = \begin{bmatrix} .7 & 0 & 0 & .4 \\ 0 & .5 & .2 & 0 \\ 0 & .5 & .8 & 0 \\ .3 & 0 & 0 & .6 \end{bmatrix}$ there will be 2 vectors since A, D cells don't interact w/ B, C cells

$P - I = \begin{bmatrix} -.3 & 0 & 0 & .4 \\ 0 & -.5 & .2 & 0 \\ 0 & .5 & -.2 & 0 \\ .3 & 0 & 0 & -.4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & -4/3 \\ 0 & 1 & -2/5 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$x_1 = 4/3 x_4$
$x_2 = 2/5 x_3$
$x_3 = x_3$
$x_4 = x_4$

$\vec{x} = \begin{bmatrix} 4 \\ 0 \\ 0 \\ 3 \end{bmatrix} x_4 + x_3 \begin{bmatrix} 0 \\ 2 \\ 5 \\ 0 \end{bmatrix}$

$x_1 + x_4 = 7$
$x_2 + x_3 = 7$

$\vec{q} = t \begin{bmatrix} 4/7 \\ 0 \\ 0 \\ 3/7 \end{bmatrix} + (1-t) \begin{bmatrix} 0 \\ 2/7 \\ 5/7 \\ 0 \end{bmatrix} \quad 0 \le t \le 1$

i. $P = \begin{bmatrix} .7 & 0 & 0 & .4 \\ .2 & .5 & .1 & 0 \\ 0 & .5 & .1 & 0 \\ .1 & 0 & .8 & .6 \end{bmatrix} \quad P - I = \begin{bmatrix} -.3 & 0 & 0 & .4 \\ .2 & -.5 & .1 & 0 \\ 0 & .5 & -.9 & 0 \\ .1 & 0 & .8 & -.4 \end{bmatrix} \Rightarrow$

$\begin{bmatrix} 1 & 0 & 0 & -4/3 \\ 0 & 1 & 0 & -3/5 \\ 0 & 0 & 1 & -1/3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$x_1 = 4/3 x_4$
$x_2 = 3/5 x_4$
$x_3 = 1/3 x_4$
$x_4 = x_4$

$\vec{x} = \begin{bmatrix} 20 \\ 9 \\ 5 \\ 15 \end{bmatrix}$

$x_1 + x_2 + x_3 + x_4 = 49$

$\vec{q} = \begin{bmatrix} 20/49 \\ 9/49 \\ 5/49 \\ 15/49 \end{bmatrix}$

One vector. enough interaction that you can go from A to B, B to C, C to D, and from D back to A.

j. $P = \begin{bmatrix} .36 & .09 & 0 & .4 \\ .2 & .5 & .2 & .13 \\ .25 & .4 & .8 & 0 \\ .19 & .01 & 0 & .47 \end{bmatrix} \quad P - I = \begin{bmatrix} -.64 & .09 & 0 & .4 \\ .2 & -.5 & .2 & .13 \\ .25 & .4 & -.2 & 0 \\ .19 & .01 & 0 & -.53 \end{bmatrix} \Rightarrow$

1j cont'd.

$$\begin{bmatrix} 1 & 0 & 0 & -11/5 \\ 0 & 1 & 0 & -56/5 \\ 0 & 0 & 1 & -503/20 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

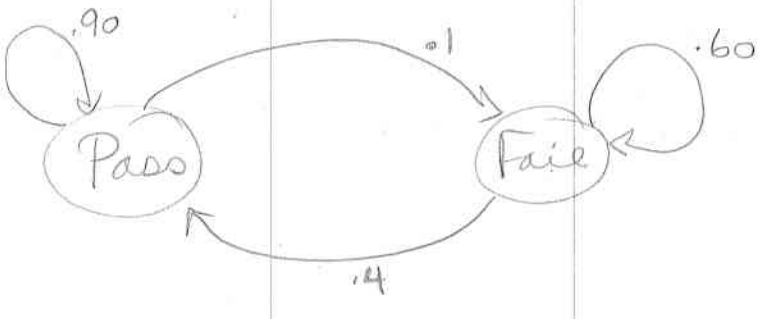
$x_1 = 11/5\, x_4$
$x_2 = 56/5\, x_4$
$x_3 = 503/20\, x_4$
$x_4 = x_4$

$$\vec{x} = \begin{bmatrix} 44 \\ 224 \\ 503 \\ 20 \end{bmatrix} \qquad x_1 + x_2 + x_3 + x_4 = 791$$

$$\vec{q} = \begin{bmatrix} 44/791 \\ 32/113 \\ 503/791 \\ 20/791 \end{bmatrix}$$

one vector, as above.

2. a.



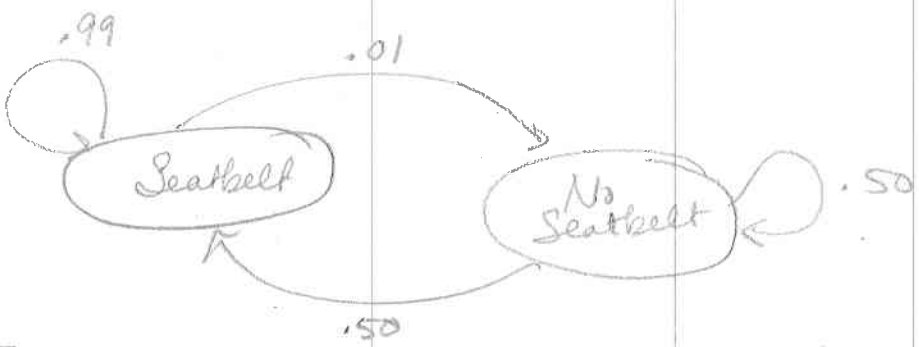
$$P = \begin{bmatrix} .9 & .6 \\ .1 & .4 \end{bmatrix} \qquad P x_0 = \begin{bmatrix} .825 \\ .175 \end{bmatrix} = x_1 \qquad x_0 = \begin{bmatrix} .75 \\ .25 \end{bmatrix}$$

82.5% will pass
17.5% will fail
Quiz 2

$$x_2 = P^2 x_0 = \begin{bmatrix} .8475 \\ .1525 \end{bmatrix}$$

84.75% pass Third quiz
15.25% fail Third quiz

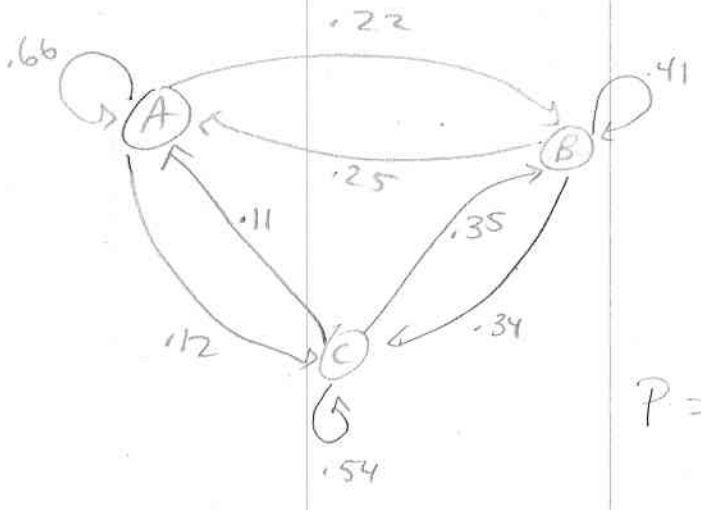
b.



$$P = \begin{bmatrix} .99 & .5 \\ .01 & .5 \end{bmatrix} \qquad P^{15} x_0 \approx \begin{bmatrix} .98 \\ .02 \end{bmatrix} \qquad x_0 = \begin{bmatrix} .60 \\ .40 \end{bmatrix}$$

$$\lim_{n \to \infty} P^n x_0 = \begin{bmatrix} 50/51 \\ 1/51 \end{bmatrix} = \vec{q}$$

2c.



$$X_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$P = \begin{bmatrix} .66 & .25 & .11 \\ .22 & .41 & .35 \\ .12 & .34 & .54 \end{bmatrix}$$

$$PX_0 = \begin{bmatrix} .66 \\ .22 \\ .12 \end{bmatrix}$$ next feeding

= 66 mice at A
22 to B
12 at C

$$P - I = \begin{bmatrix} -.34 & .25 & .11 \\ .22 & -.59 & .35 \\ .12 & .34 & -.46 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & -381/364 \\ 0 & 1 & -179/182 \\ 0 & 0 & 0 \end{bmatrix}$$

$$X_1 = 381/364\, X_3$$
$$X_2 = 179/182\, X_3$$
$$X_3 = X_3$$

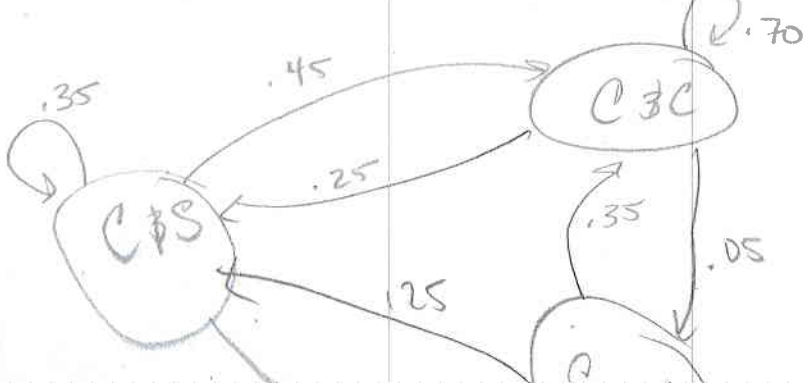
$$\vec{X} = \begin{bmatrix} 381 \\ 358 \\ 364 \end{bmatrix} \qquad X_1 + X_2 + X_3 = 1103$$

$$\vec{q} = \begin{bmatrix} 381/1103 \\ 358/1103 \\ 364/1103 \end{bmatrix} \approx \begin{bmatrix} .345 \\ .3246 \\ .33 \end{bmatrix}$$

35 mice at A
32 mice at B
33 mice at C

This is all the mice

d.



$$P = \begin{bmatrix} .35 & .25 & .25 \\ .45 & .70 & .35 \\ .2 & .05 & .40 \end{bmatrix}$$

d. cont'd

$$P - I = \begin{bmatrix} -.65 & .25 & .25 \\ .45 & -.3 & .35 \\ .2 & .05 & -.6 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & -65/33 \\ 0 & 1 & -136/33 \\ 0 & 0 & 0 \end{bmatrix}$$

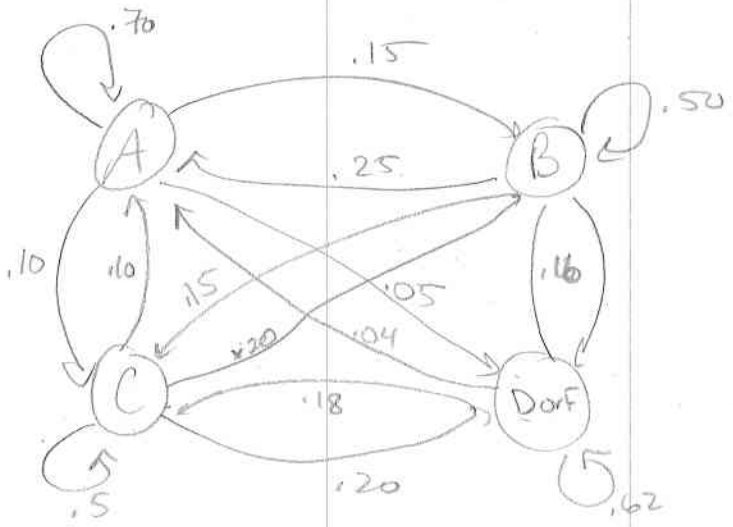
$$x_1 = 65/33\, x_3$$
$$x_2 = 136/33\, x_3$$
$$x_3 = x_3$$

$$\Rightarrow \vec{x} = \begin{bmatrix} 65 \\ 136 \\ 33 \end{bmatrix} \qquad x_1 + x_2 + x_3 = 231 \qquad \vec{q} = \begin{bmatrix} 65/231 \\ 136/231 \\ 33/231 \end{bmatrix}$$

$$\approx \begin{bmatrix} .28 \\ .59 \\ .14 \end{bmatrix}$$

28% of days cold and sunny
59% cold and cloudy
and 14% have snow

e.



$$X_0 = \begin{bmatrix} .30 \\ .35 \\ .25 \\ .10 \end{bmatrix}$$

$$P = \begin{bmatrix} .70 & .25 & .10 & .04 \\ .15 & .50 & .20 & .16 \\ .10 & .15 & .50 & .18 \\ .05 & .1 & .20 & .62 \end{bmatrix}$$

$$P^8 X_0 \approx \begin{bmatrix} .31 \\ .25 \\ .22 \\ .22 \end{bmatrix}$$

$$P - I = \begin{bmatrix} -.30 & .25 & .10 & .04 \\ .15 & -.5 & .2 & .16 \\ .10 & .15 & -.5 & .18 \\ .05 & .1 & .20 & -.38 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & -246/175 \\ 0 & 1 & 0 & -397/350 \\ 0 & 0 & 1 & -687/700 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

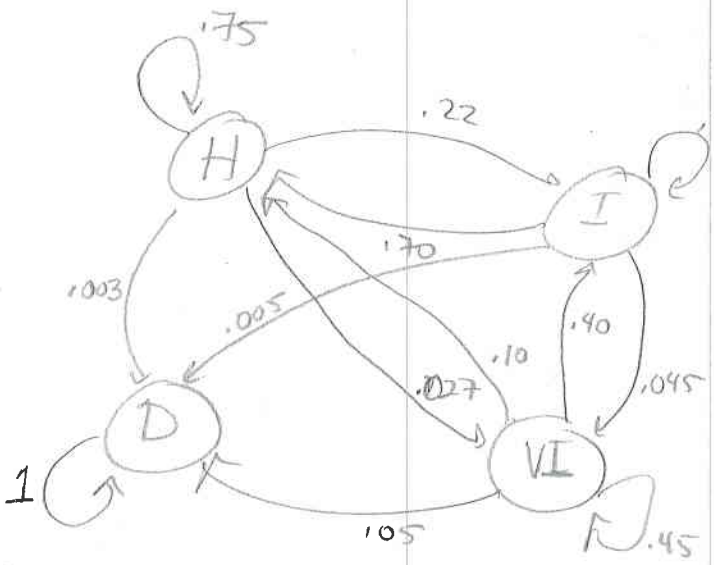
$$x_1 = 246/175\, x_4$$
$$x_2 = 397/350\, x_4$$
$$x_3 = 687/700\, x_4$$
$$x_4 = x_4$$

$$\vec{x} = \begin{bmatrix} 984 \\ 594 \\ 687 \\ 700 \end{bmatrix} \qquad x_1 + x_2 + x_3 + x_4 = 2965$$

$$\vec{q} = \begin{bmatrix} 984/2965 \\ 594/2965 \\ 687/2965 \\ 700/2965 \end{bmatrix} \approx \begin{bmatrix} .33 \\ .20 \\ .23 \\ .24 \end{bmatrix}$$

33% get an A, 20% get a B, 23% get a C and 24% get a D or worse

2f.



$$X_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$P = \begin{bmatrix} .75 & .70 & .10 & 0 \\ .22 & .25 & .40 & 0 \\ .027 & .045 & .45 & 0 \\ .003 & .005 & .05 & 1 \end{bmatrix}$$

$$PX_0 = \begin{bmatrix} .75 \\ .22 \\ .027 \\ .003 \end{bmatrix}$$

750 people are well, 220 ill, 27 very ill and 3 have died

it takes ≥ 1000 steps

Somewhere around 1300-1400 steps is the minimum

$\approx$ 108 years