Math 2568, Exam #3 – Part I, Fall 2013 Name KEY

Instructions: Show all work. Use exact answers unless specifically asked to round. Justify answers will work or you may receive no credit. You may **not** use a calculator on this portion of the exam.

1. Suppose matrix A is a 9x5 matrix with 4 pivot columns. Determine the following. (10 points)

dim Col A = 4 dim Nul A = 1

dim Row A^T = 4 If Col A is a subspace of $\mathbb{R}^m$, then m = 9

Rank A = 4 If Nul A is a subspace of $\mathbb{R}^n$, then n = 5

2. Consider the stochastic Markov chain matrix given by the matrix $A = \begin{bmatrix} .7 & .2 \\ .3 & .8 \end{bmatrix}$. Calculate the equilibrium vector of the system. (5 points)

$P - I = \begin{bmatrix} -.3 & .2 \\ .3 & -.2 \end{bmatrix}$ $.3x_1 = .2x_2 \Rightarrow$ $\begin{array}{l} x_1 = \frac{2}{3}x_2 \\ x_2 = x_2 \end{array}$ $\Rightarrow \vec{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

$q = \begin{bmatrix} 2/5 \\ 3/5 \end{bmatrix} = \begin{bmatrix} .4 \\ .6 \end{bmatrix}$

3. List at least 8 properties of Invertible Matrices from the Invertible Matrix Theorem. (8 points)

There are 20 to choose from. Answers will vary.

1) A is invertible
2) A^T is invertible
3) det $A \neq 0$
4) Nul $A = \{0\}$
5) A is one-to-one
6) A is onto
7) Col A is basis for $\mathbb{R}^n$.
8) dim Nul A = 0

4. Find the eigenvalues and eigenvectors of the matrices below. Be sure to clearly indicate the characteristic equation, and which eigenvalues and eigenvectors go together. (10 points)

a. $A = \begin{bmatrix} 7 & 2 \\ 4 & 5 \end{bmatrix}$

$$\det \begin{bmatrix} 7-\lambda & 2 \\ 4 & 5-\lambda \end{bmatrix} = (7-\lambda)(5-\lambda) - 8 = \lambda^2 - 12\lambda + 27 = 0$$

$$(\lambda - 9)(\lambda - 3) = 0$$

$\lambda_1 = 9, \ \lambda_2 = 3$

$$\begin{bmatrix} -2 & 2 \\ 4 & -4 \end{bmatrix} \Rightarrow -2x_1 = -2x_2 \Rightarrow \begin{matrix} x_1 = x_2 \\ x_2 = x_2 \end{matrix} \Rightarrow \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 2 \\ 4 & 2 \end{bmatrix} \Rightarrow 4x_1 = -2x_2 \Rightarrow \begin{matrix} x_1 = -\frac{1}{2}x_2 \\ x_2 = x_2 \end{matrix} \Rightarrow \vec{v}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

b. $B = \begin{bmatrix} 7 & -20 \\ 4 & -1 \end{bmatrix}$

$$\det \begin{bmatrix} 7-\lambda & -20 \\ 4 & -1-\lambda \end{bmatrix} \Rightarrow (7-\lambda)(-1-\lambda) + 80 \Rightarrow \lambda^2 - 6\lambda + 73 = 0$$

$$\lambda = \frac{6 \pm \sqrt{36 - 4(73)}}{2} = \frac{6 \pm \overset{16i}{\sqrt{-256}}}{2} = 3 \pm 8i$$

$\lambda_1 = 3 + 8i, \ \lambda_2 = 3 - 8i$

$$\begin{bmatrix} 7-(3+8i) & -20 \\ 4 & -1-(3+8i) \end{bmatrix} = \begin{bmatrix} 4-8i & -20 \\ 4 & -4-8i \end{bmatrix}$$

$$\frac{4x_1}{4} = \frac{(4+8i)x_2}{4}$$

$$x_1 = (1+2i)x_2$$
$$x_2 = x_2$$

$$v_1 = \begin{bmatrix} 1+2i \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} i$$

$$v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} i$$

5. For the matrix $A = \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}$, with eigenvalues $\lambda_1 = 3, \lambda_2 = 2$ and eigenvectors $\vec{v_1} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \vec{v_2} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, find a similarity transformation matrix P so that A can be diagonalized. Clearly state both P and D. (5 points)

$P = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$ $D = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$

6. Given the vectors $\vec{u} = \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix}, \vec{v} = \begin{bmatrix} -2 \\ 6 \\ 3 \end{bmatrix}$ find the following.

a. $\vec{v} \cdot \vec{u}$ (2 points)

$-6 - 6 + 12 = 0$

b. $||\vec{u}||$. (2 points)

$\sqrt{9+1+16} = \sqrt{26}$

c. A unit vector in the direction of $\vec{u}$. (2 points)

$\frac{\vec{u}}{||\vec{u}||} = \hat{u} = \begin{bmatrix} 3/\sqrt{26} \\ -1/\sqrt{26} \\ 4/\sqrt{26} \end{bmatrix}$

d. Find the distance between $\vec{u}$ and $\vec{v}$. (3 points)

$\vec{u} - \vec{v} = \begin{bmatrix} 5 \\ -7 \\ 1 \end{bmatrix}$ $||\vec{u} - \vec{v}|| = \sqrt{25+49+1} = \sqrt{75} = 5\sqrt{3}$

e. Are $\vec{u}$ and $\vec{v}$ orthogonal? Why or why not? (2 points)

yes. since the dot product is zero.

7. Determine if each statement is True or False. (2 points each)

a. T F Every eigenvalue has only one corresponding eigenvector.

b. T F An nxn matrix will always have exactly n real eigenvalues.

c. T F If A and B are row equivalent, then their column spaces are the same.

d. T F $P_{C\leftarrow B} = P_B^{-1}P_C$

e. T F A linearly independent set that spans the space in a subspace H is a basis for H.

f. T F If the steady-state vector for a stochastic matrix is unique then the Markov Chain matrix has no absorbing states and has communication between all available states.

g. T F A matrix is invertible if and only if 0 is not an eigenvalue of A.

h. T F The eigenvalues of a matrix are always on its main diagonal.

i. T F The eigenspace of an nxn matrix with n distinct real eigenvalues always form a basis for R^n.

j. T F A trajectory of a dynamical system is a set of ordered vectors $\overrightarrow{x_k}$ that tracks the population values of a system over time.

k. T F The elementary row operations of A do not change its eigenvalues.

l. T F If A is diagonalizable, then A is invertible.

m. T F The complex eigenvalues of a discrete dynamical system either both attract to the origin or both repel from the origin.

Math 2568, Exam #3 – Part II, Fall 2013 Name ______________________

Instructions: Show all work. Use exact answers unless specifically asked to round. Justify answers will work or you may receive no credit. You may use a calculator on this portion of the exam.

1. a. For the matrix $B = \begin{bmatrix} 2 & 2 \\ -13 & -8 \end{bmatrix}$, with eigenvalues $\lambda = -3 \pm i$, with eigenvectors $\vec{v} = \begin{bmatrix} -5 \\ 13 \end{bmatrix} + \begin{bmatrix} \mp 1 \\ 0 \end{bmatrix} i$. Find one similarity transformation P that will transform B=PCP^{-1}, where C is a scaled rotation matrix. State both P and C. (5 points)

$$P = \begin{bmatrix} -5 & -1 \\ 13 & 0 \end{bmatrix} \qquad C = \begin{bmatrix} -3 & -1 \\ 1 & -3 \end{bmatrix}$$

b. Use the C matrix from part a, and find the scaling factor and then calculate the angle of rotation of the matrix. Round your angle to 3 decimal places in radians, or to the nearest whole degree. (5 points)

$$C = \begin{bmatrix} r\cos\theta & -r\sin\theta \\ r\sin\theta & r\cos\theta \end{bmatrix} \qquad \sqrt{3^2+1^2} = \sqrt{10} = r$$

$$\sqrt{10}\begin{bmatrix} -3/\sqrt{10} & -1/\sqrt{10} \\ 1/\sqrt{10} & -3/\sqrt{10} \end{bmatrix} \qquad \cos^{-1}(-3/\sqrt{10}) = \theta = 2.8198..$$

$\uparrow$

Cos +
Sin − $\Rightarrow$ Q II

$\theta = 161.56°$

2. Assume that $A=\begin{bmatrix} 1 & 2 & -1 & 2 & 0 & 3 \\ 3 & 4 & -1 & 1 & 2 & 5 \\ 1 & 2 & 0 & 4 & 1 & -8 \\ 2 & 6 & 1 & 8 & 1 & 11 \end{bmatrix}$.

a. Find a basis for the null space of A. (6 points)

$$\text{rref} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 21/11 & -170/11 \\ 0 & 1 & 0 & 0 & -9/11 & 153/11 \\ 0 & 0 & 1 & 0 & 7/11 & -9/11 \\ 0 & 0 & 0 & 1 & 2/11 & -56/11 \end{bmatrix}$$

$$\begin{aligned} x_1 &= -21/11\, x_5 + 170/11\, x_6 \\ x_2 &= 9/11\, x_5 - 153/11\, x_6 \\ x_3 &= -7/11\, x_5 + 9/11\, x_6 \\ x_4 &= -2/11\, x_5 + 56/11\, x_6 \\ x_5 &= x_5 \\ x_6 &= x_6 \end{aligned} \Rightarrow \vec{x} = t\begin{bmatrix} -21 \\ 9 \\ -7 \\ -2 \\ 11 \\ 0 \end{bmatrix} + s\begin{bmatrix} 170 \\ -153 \\ 9 \\ 56 \\ 0 \\ 11 \end{bmatrix}$$

b. Find the dimension of the kernel? (3 points)

2

3. Given the bases $B = \{1 + 3t - t^2, 2 + 5t - 2t^2, 7 - t + 4t^2\}$ and $C = \{2 - 3t + 2t^2, 1 - 4t + 3t^2, 6t + t^2\}$ below, find the change of basis matrices $\underset{C \leftarrow B}{P}$ and $\underset{B \leftarrow C}{P}$. For the B-coordinate vector $\vec{p}$ given as $\left[\vec{p}\right]_B = \begin{bmatrix} 1 \\ 6 \\ -2 \end{bmatrix}$, find the C-coordinate vector for $\vec{p}$, and find the original $p(t)$ in the standard basis. Be sure to state any matrices you use to solve.(8 points)

$$P_B = \begin{bmatrix} 1 & 2 & 7 \\ 3 & 5 & -1 \\ -1 & -2 & 4 \end{bmatrix} \quad P_C = \begin{bmatrix} 2 & 1 & 0 \\ -3 & -4 & 6 \\ 2 & 3 & 1 \end{bmatrix} \quad P_{C \leftarrow B} = P_C^{-1} P_B = \begin{bmatrix} 31/29 & 61/29 & 129/29 \\ -33/29 & -64/29 & -55/29 \\ 8/29 & 12/29 & 23/29 \end{bmatrix}$$

$$P_{B \leftarrow C} = \begin{bmatrix} -28/11 & 5/11 & 169/11 \\ 1 & -1 & -8 \\ 4/11 & 4/11 & 1/11 \end{bmatrix} \quad P_{C \leftarrow B}[\vec{p}]_B = \begin{bmatrix} 139/29 \\ -307/29 \\ 34/29 \end{bmatrix} = [\vec{p}]_C$$

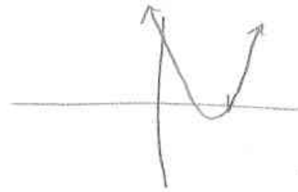
$$P_B [\vec{p}]_B = \begin{bmatrix} -1 \\ 35 \\ -21 \end{bmatrix} \Rightarrow p(t) = -1 + 35t - 21t^2$$

4. Consider the discrete dynamical system given by the matrix $A=\begin{bmatrix} .4 & .15 \\ -.7 & 1.2 \end{bmatrix}$.

a. Determine the behaviour of the origin for this system: is it a repeller, an attractor or a saddle point? (7 points)

$(.4-\lambda)(1.2-\lambda)+(.7)(.15) \Rightarrow \lambda_1 = .5654 \quad \lambda_2 = 1.0345$

Saddle point



b. Given the initial condition of the population as $x_0=\begin{bmatrix}5\\12\end{bmatrix}$, find 10 points of the trajectory for the system and list them here. (5 points)

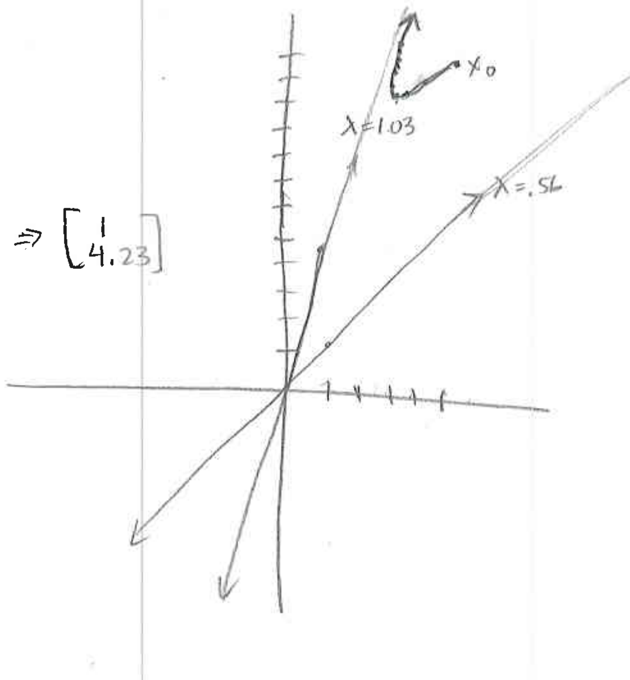
$\begin{bmatrix}5\\12\end{bmatrix}, \begin{bmatrix}3.8\\10.9\end{bmatrix}, \begin{bmatrix}3.155\\10.42\end{bmatrix}, \begin{bmatrix}2.825\\10.29\end{bmatrix}, \begin{bmatrix}2.67\\10.37\end{bmatrix}, \begin{bmatrix}2.62\\10.58\end{bmatrix},$

$\begin{bmatrix}2.63\\10.85\end{bmatrix}, \begin{bmatrix}2.68\\11.18\end{bmatrix}, \begin{bmatrix}2.75\\11.54\end{bmatrix}, \begin{bmatrix}2.83\\11.92\end{bmatrix}, \begin{bmatrix}2.92\\12.32\end{bmatrix}$

c. Plot the points on a graph together with the eigenvectors of the system. Make sure your graph is big enough to clearly read it. Connect the trajectory with a curve and an arrow indicating the flow of time. (8 points)

$\begin{bmatrix}-.1654 & .15\\ -.7 & .6346\end{bmatrix} \quad x_1 = \frac{-.15}{-.1654}x_2 \Rightarrow \begin{bmatrix}1\\1.1\end{bmatrix}$

$\begin{bmatrix}-.6345 & .15\\ -.7 & .1655\end{bmatrix} \quad x_1 = \frac{-.15}{-.6345}x_2 \Rightarrow \begin{bmatrix}.236\\1\end{bmatrix} \Rightarrow \begin{bmatrix}1\\4.23\end{bmatrix}$



5. Determine if the functions $f(x) = \sin(x)$ and $g(x) = \cos(x)$ are orthogonal under the inner product $\langle f, g \rangle = \int_{-\pi}^{\pi} f(x)g(x)dx$. (6 points)

$$\int_{-\pi}^{\pi} \sin x \cos x \, dx = \frac{1}{2}\sin^2 x \Big|_{-\pi}^{\pi} = 0$$

yes, they are orthogonal.

6. Answer each of the equations below as completely as possible. (5 points each)

   a. How does one determine the dimension of a vector space (or subspace)?

   the dimension of the vector space is determined by the # of vectors in the basis.

   b. Explain why the equilibrium vector of a stochastic matrix must correspond to an eigenvalue of one.

   $Pq = q$ is equivalent to $A\vec{v} = \lambda\vec{v}$

   where λ is 1 and $\vec{v} = q$

   Since the vector produces itself rather than a multiple of itself the eigenvalue must be one.

c. Explain the difference between "orthogonal" and "perpendicular".

perpendicular is a geometric property. The usual dot product in $\mathbb{R}^n$ makes orthogonality equivalent to perpendicularity. But, orthogonality depends on the inner product used and can be extended to non-geometric properties and spaces.