# Coordinate Systems

When we work with vectors, we sometimes end up so accustomed to the standard way of presenting vectors that we forget what the shorthand is really telling us, and so it can be confusing when we start thinking of vectors when they are built in different coordinate systems. This handout will break down how the standard representation of vectors works, primarily in two dimensions for the sake of representing things graphically, and we will then extend the representation to non-standard bases and how to do the conversions. These processes will have some things in common, but some of them will depend on the properties of the building blocks we will be working with.

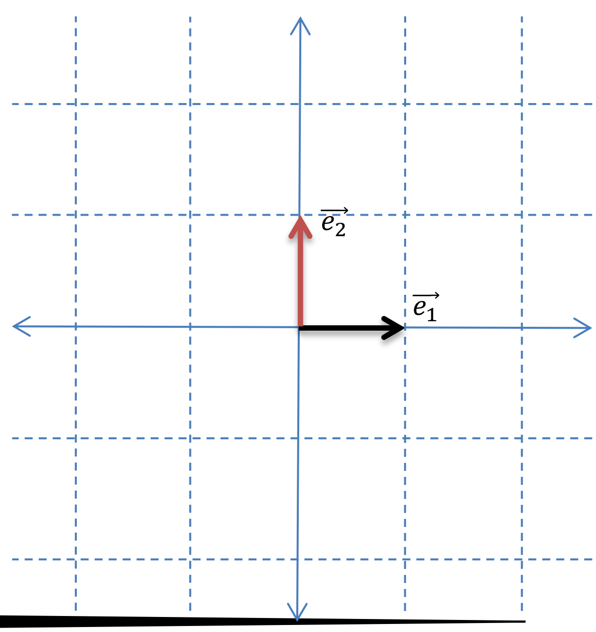
Let's begin with some basic terminology and notations.

A ***basis*** is a set of vectors from which a coordinate system is built. Every basis has enough vectors to span the entire vector space it represents, but not so many that the set becomes linearly dependent. The number of vectors in the basis is the standard way of talking about the ***dimension*** of the space. So a space like $\mathbb{R}^2$ will always have two vectors in any set of basis vectors. $\mathbb{R}^3$ will always have three vectors in any basis, and so forth.

The ***set of standard basis vectors*** is notated by $\mathscr{E}$, and the elements, the standard basis vectors themselves, are numbered $\vec{e_i}$. These vectors are represented by a vector with a 1 in one coordinate and 0 in all others. The location of the 1 matches the subscript. So in $\mathbb{R}^2$, $\vec{e_1} = \begin{bmatrix}1\\0\end{bmatrix}, \vec{e_2} = \begin{bmatrix}0\\1\end{bmatrix}$. In $\mathbb{R}^5$, there are five basis vectors given by: $\vec{e_1} = \begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix}, \vec{e_2} = \begin{bmatrix}0\\1\\0\\0\\0\end{bmatrix}, \vec{e_3} = \begin{bmatrix}0\\0\\1\\0\\0\end{bmatrix}, \vec{e_4} = \begin{bmatrix}0\\0\\0\\1\\0\end{bmatrix}, \vec{e_5} = \begin{bmatrix}0\\0\\0\\0\\1\end{bmatrix}$.

**1. Geometric Interpretation.**

Let's think for a minute about what these vectors represented geometrically in $\mathbb{R}^2$. The $\vec{e_1}$ vector is a unit vector (with length one) in the direction of the x-axis; it has no y-component. The $\vec{e_2}$ vector is a unit vector (with length one) in the direction of the y-axis; it has no x-component. I've drawn these vectors on the graph below.



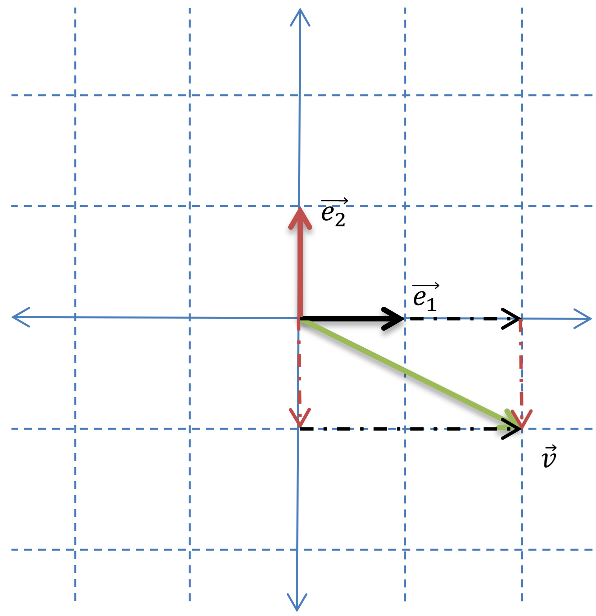

If we want to think about what a coordinate vector represents, it's a linear combination of these basis vectors. To put it another way, it's some multiple of the first basis vector added to some multiple of the second basis vector.

So if we think about what the vector $\vec{v} = \begin{bmatrix}2\\-1\end{bmatrix}$ means, it's $\begin{bmatrix}2\\-1\end{bmatrix} = 2\begin{bmatrix}1\\0\end{bmatrix} + (-1)\begin{bmatrix}0\\1\end{bmatrix} = 2\vec{e_1} + (-1)\vec{e_2}$. The resulting vector is graphed below.

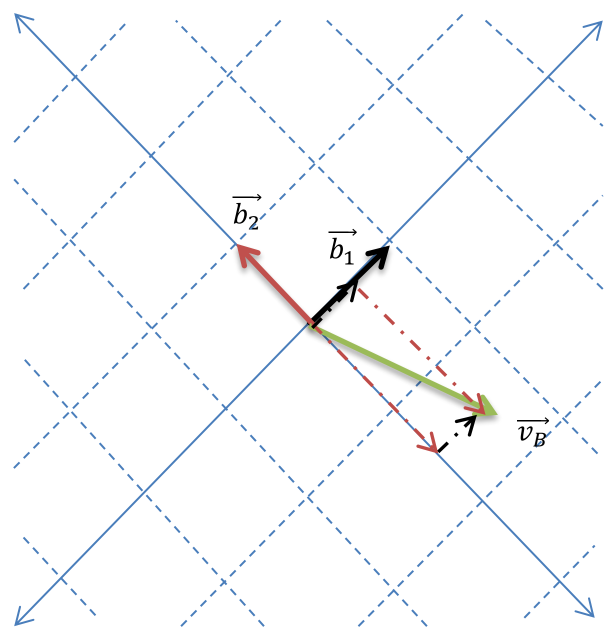
So for any vector $\vec{x} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = c_1\vec{b_1} + c_2\vec{b_2}$, where the vectors $\vec{b_\iota}$ are the basis vectors for the space.

If we change the basis vectors and use something other than the standard basis vectors, the coordinate representation of the vector changes based on the new basis. The coordinates are the multiples of the basis vectors needed to arrive at the given location in space.

Part of the difficulty here is that to understand what is going on, we have to compare what we are doing in the new basis to what is going on in the standard basis. The standard basis feels so obvious to us that extending it to a general case can seem challenging.




**Example 1**. Let us suppose that we now change our basis vectors to $\mathscr{B}$ given by $\vec{b_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \vec{b_2} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ instead of the standard basis vectors. These basis vectors have the property of essentially rotating our basis vectors by 45°. Consider the graph below. The grid lines shown are at whole number multiples of the basis vectors, and in the direction of the basis vectors. But I didn't move the vector $\vec{v}$, here noted as $\vec{v_B}$. Notice that I now need different multiples of the new basis vectors in order to get to the same exact point in the plane. We can represent that as $c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, or that $\vec{v_B} = [\vec{v}]_B = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$. The coordinate representation then in any basis are the multiples of the basis vectors needed to locate the point in space.




You may recognize this as the vector form of a system of equations which we can solve by row-reduction, or by using an inverse matrix. We find the solution is the representation of the vector in the new basis or $[\vec{v}]_B = \begin{bmatrix} \frac{1}{2} \\ -\frac{3}{2} \end{bmatrix}_B$. We can see from the graph that these values are reasonable. The component in the $\vec{b_1}$ direction is less than one full length and positive; and the value in the $\vec{b_2}$ direction is negative, and more than one full length but less than twice using the parallelogram rule of vector addition.

The example alternative basis $\mathscr{B}$ used above contained two vectors that were also perpendicular to each other, as the standard basis is, but the vectors are a bit longer since the vectors are not of length one. We could rescale these vectors to normalize them to $\overrightarrow{b_1} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \overrightarrow{b_2} = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$, but rescaling them like this would also rescale the coordinates of the point in the opposite direction (the coordinates would need to be larger since we scaled the basis vectors to be shorter).

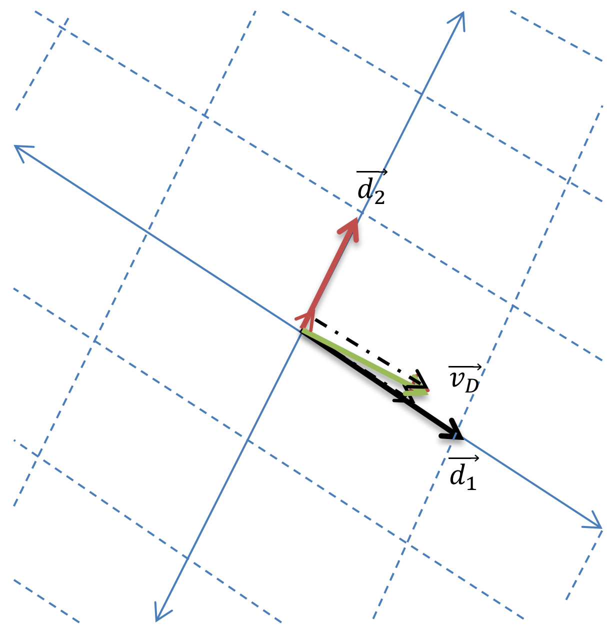
In addition to using a perpendicular basis like this, we could also use vectors at other than a 90° angle to each other. Our grid lines won't cross in squares or rectangles by rhombuses or parallelograms, but in principle, the concept is the same.

**Example 2**. Let's consider the $\mathscr{D}$ basis given by $\overrightarrow{d_1} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}, \overrightarrow{d_2} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. The basis vectors and our same $\vec{v}$ as before are graphed below.

Here, our vector is almost entirely composed of a scalar multiples of our first basis vector (but shorter than the basis vector), and a small bit of the second basis vector, with both of the components positive. So let's solve the system and see what representation we get.

$$c_1 \begin{bmatrix} 3 \\ -2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$[\vec{v}]_D = \begin{bmatrix} \frac{5}{8} \\ \frac{1}{8} \end{bmatrix}_D$$




We can always verify that our calculations are correct because if we use these coefficients with the basis vectors, they will simplify to the original vector in the standard basis.

$$\frac{5}{8}\begin{bmatrix} 3 \\ -2 \end{bmatrix} + \frac{1}{8}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{15}{8} + \frac{1}{8} \\ -\frac{10}{8} + \frac{2}{8} \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

**Practice Problems.**

a. Find the representation for the $\vec{x} = \begin{bmatrix} -4 \\ -3 \end{bmatrix}$ in each specified basis. Draw each example graphically to check your work.

   i. $\mathscr{B} = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \end{bmatrix} \right\}$

   ii. $\mathscr{C} = \left\{ \begin{bmatrix} -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$

   iii. $\mathscr{D} = \left\{ \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix} \right\}$

iv. $\mathscr{T} = \left\{ \begin{bmatrix} 7 \\ -9 \end{bmatrix}, \begin{bmatrix} -3 \\ -1 \end{bmatrix} \right\}$

b. Find the representation of the following vectors in the basis $\mathscr{G} = \left\{ \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$.

i. $\vec{x} = \begin{bmatrix} 7 \\ 8 \end{bmatrix}$

ii. $\vec{y} = \begin{bmatrix} -5 \\ 6 \end{bmatrix}$

iii. $\vec{z} = \begin{bmatrix} 11 \\ 2 \end{bmatrix}$

This is the geometric interpretation of coordinates in bases other than the standard one. Solving for the coordinate values in different bases have a number of different techniques. We are going to discuss three situations:

1) Converting between the standard basis and a single non-standard one (or vice versa).
2) Converting between two different non-standard bases
3) The special case of converting between the standard basis and an orthogonal basis.

We will take each situation in turn.

**2. One non-standard basis.**

We can look back at our system of equation solved in the geometric discussions above to develop the general case. Here, we said that $c_1\overrightarrow{b_1} + c_2\overrightarrow{b_2} = \vec{x}$, where $\overrightarrow{b_1}, \overrightarrow{b_2}$ are the basis vectors, $c_1, c_2$ are the coordinates of the vector in the new basis, and $\vec{x}$ is the representation of the vector in the standard basis (for $\mathbb{R}^2$ here). We can rewrite this vector equation in matrix form as $\begin{bmatrix} \overrightarrow{b_1} & \overrightarrow{b_2} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \vec{x}$, where $\begin{bmatrix} \overrightarrow{b_1} & \overrightarrow{b_2} \end{bmatrix}$ is a matrix with the basis vectors as the columns of the matrix. We call this matrix $P_B$, and $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = [\vec{x}]_B$, the representation of the vector in the new basis.

$$P_B[\vec{x}]_B = \vec{x}$$

We can generalize this to vector spaces of any size with the appropriate size matrix (which will always be square and invertible since the basis vectors must be both independent and span the space). Thus, if we are given a vector in the standard basis, and want to solve for the coordinate representation of the vector in the new basis, we can solve this equation or use the formula.

$$[\vec{x}]_B = P_B^{-1}\vec{x}$$

**Example 3**. Let's consider $\mathscr{T} = \left\{ \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 5 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} \right\}$, and the vector $\vec{x} = \begin{bmatrix} -5 \\ -2 \\ 15 \end{bmatrix}$. Find the representation of this vector $[\vec{x}]_B$ in the new basis $\mathscr{T}$.

First, we need to construct the $P_F$ matrix. $P_F = \begin{bmatrix} 1 & 5 & 2 \\ 3 & 0 & -2 \\ 4 & -1 & 1 \end{bmatrix}$. Then find the inverse: $P_F^{-1} = \begin{bmatrix} \frac{2}{63} & \frac{1}{9} & \frac{10}{63} \\ \frac{11}{63} & \frac{1}{9} & -\frac{8}{63} \\ \frac{1}{21} & -\frac{1}{3} & \frac{5}{21} \end{bmatrix}$. And then multiply $P_F^{-1}\vec{x} = \begin{bmatrix} \frac{2}{63} & \frac{1}{9} & \frac{10}{63} \\ \frac{11}{63} & \frac{1}{9} & -\frac{8}{63} \\ \frac{1}{21} & -\frac{1}{3} & \frac{5}{21} \end{bmatrix} \begin{bmatrix} -5 \\ -2 \\ 15 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix}_F = [\vec{x}]_F$.

We can check our work to be sure: $2\begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} - 3\begin{bmatrix} 5 \\ 0 \\ -1 \end{bmatrix} + 4\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2-15+8 \\ 6-0-8 \\ 8+3+4 \end{bmatrix} = \begin{bmatrix} -5 \\ -2 \\ 15 \end{bmatrix}$.

**Practice Problems.**

c. Find the $P_B$ matrix (or $P_H$, $P_J$, and $P_K$ in each case), and the representation of the give vector in the specified basis using the method in Example 3.
   i. $\mathscr{H} = \left\{ \begin{bmatrix} 1 \\ -4 \end{bmatrix}, \begin{bmatrix} -5 \\ 2 \end{bmatrix} \right\}, \vec{x} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$
   ii. $\mathscr{J} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -5 \end{bmatrix} \right\}, \vec{x} = \begin{bmatrix} 9 \\ -3 \\ 1 \end{bmatrix}$
   iii. $\mathscr{K} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 8 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ -1 \\ -2 \end{bmatrix} \right\}, \vec{x} = \begin{bmatrix} 10 \\ -5 \\ 4 \\ 6 \end{bmatrix}$

d. Find the matrix $P_B$ (or $P_L$, $P_M$, and $P_N$ in each case) and the vector in the standard basis given the basis and the representation of the matrix in that basis.
   i. $\mathscr{L} = \left\{ \begin{bmatrix} 1 \\ -4 \end{bmatrix}, \begin{bmatrix} -5 \\ 2 \end{bmatrix} \right\}, [\vec{x}]_L = \begin{bmatrix} 3 \\ 3 \end{bmatrix}_L$
   ii. $\mathscr{M} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -5 \end{bmatrix} \right\}, [\vec{x}]_M = \begin{bmatrix} 9 \\ -3 \\ 1 \end{bmatrix}_M$
   iii. $\mathscr{N} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 8 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ -1 \\ -2 \end{bmatrix} \right\}, [\vec{x}]_N = \begin{bmatrix} 10 \\ -5 \\ 4 \\ 6 \end{bmatrix}_N$

**3. Two non-standard bases.**

Suppose we have two bases, let's call them $\mathscr{B}$ and $\mathscr{C}$. Then two equations are true from §2:

$$P_B[\vec{x}]_B = \vec{x} \quad \& \quad P_C[\vec{x}]_C = \vec{x}$$

Since we have two things equal to $\vec{x}$, we can set the two left sides equal to each other.

$$P_B[\vec{x}]_B = P_C[\vec{x}]_C$$

Solving for $[\vec{x}]_C$ we get the equation:

$$P_C^{-1} P_B [\vec{x}]_B = [\vec{x}]_C$$

Since the product of two matrices is just a matrix, we can call this matrix something special. We'll use the notation $P_C^{-1} P_B = P_{C \leftarrow B}$ (note the direction of the arrow) giving us $P_{C \leftarrow B} [\vec{x}]_B = [\vec{x}]_C$. This matrix converts a vector in the $\mathscr{B}$ basis, through multiplication, into a vector in the $\mathscr{C}$ basis.

Alternatively, if we solve for $[\vec{x}]_B$, we get:

$$P_B^{-1} P_C [\vec{x}]_C = [\vec{x}]_B \quad or \quad P_{B \leftarrow C} [\vec{x}]_C = [\vec{x}]_B$$

And so in parallel with above, we have that $P_B^{-1} P_C = P_{B \leftarrow C}$, or that $(P_{C \leftarrow B})^{-1} = P_{B \leftarrow C}$. This is the matrix that converts a vector in the $\mathscr{C}$ basis into a vector in the $\mathscr{B}$ basis.

**Example 4**. Suppose we have the basis $\mathscr{F} = \left\{ \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 5 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} \right\}$, and the basis $\mathscr{G} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$. We are given the representation of a vector $\vec{x}$ in the basis $\mathscr{F}$ as $[\vec{x}]_F = \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix}_F$. Find the representation of the vector in the G basis.

We wish to convert from F to G, so we need the matrix $P_{G \leftarrow F}$. The order of the notation reminds us the order we need to multiply the matrices in: $P_{G \leftarrow F} = P_G^{-1} P_F$.

$$P_F = \begin{bmatrix} 1 & 5 & 2 \\ 3 & 0 & -2 \\ 4 & -1 & 1 \end{bmatrix}$$

$$P_G = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 0 \\ 2 & 3 & 1 \end{bmatrix}$$

$$P_G^{-1} = \begin{bmatrix} -1 & -\frac{1}{2} & 1 \\ 0 & \frac{1}{2} & 0 \\ 2 & -\frac{1}{2} & -1 \end{bmatrix}$$

$$P_{G \leftarrow F} = P_G^{-1} P_F = \begin{bmatrix} -1 & -\frac{1}{2} & 1 \\ 0 & \frac{1}{2} & 0 \\ 2 & -\frac{1}{2} & -1 \end{bmatrix} \begin{bmatrix} 1 & 5 & 2 \\ 3 & 0 & -2 \\ 4 & -1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & -6 & 0 \\ \frac{3}{2} & 0 & -1 \\ -\frac{7}{2} & 11 & 4 \end{bmatrix}$$

$$P_{G \leftarrow F} [\vec{x}]_F = \begin{bmatrix} \frac{3}{2} & -6 & 0 \\ \frac{3}{2} & 0 & -1 \\ -\frac{7}{2} & 11 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix}_F = \begin{bmatrix} 21 \\ -1 \\ -24 \end{bmatrix}_G$$

But is this right? Do they represent the same vector in each basis? Let's check.

$$21\begin{bmatrix}1\\0\\2\end{bmatrix}+(-1)\begin{bmatrix}2\\2\\3\end{bmatrix}+(-24)\begin{bmatrix}1\\0\\1\end{bmatrix}=\begin{bmatrix}21-2-24\\0-2-0\\42-3-24\end{bmatrix}=\begin{bmatrix}-5\\-2\\15\end{bmatrix}$$

$$2\begin{bmatrix}1\\3\\4\end{bmatrix}+(-3)\begin{bmatrix}5\\0\\-1\end{bmatrix}+4\begin{bmatrix}2\\-2\\1\end{bmatrix}=\begin{bmatrix}-5\\-2\\15\end{bmatrix}$$

You may notice this is the same vector from Example 3.

One question that might concern us here is *what do the vectors in the $P_{G\leftarrow F}$ matrix represent*? These are the basis vectors of $\mathscr{F}$ represented in the basis $\mathscr{G}$ or $[\vec{f_i}]_G$. Let's check one of them, just to see.

$$\vec{f_1}=\begin{bmatrix}1\\3\\4\end{bmatrix}, P_G^{-1}\vec{f_1}=\begin{bmatrix}\frac{3}{2}\\\frac{3}{2}\\-\frac{7}{2}\end{bmatrix}_G=[\vec{f_1}]_G$$

**Practice Problems.**

e. Given the bases B and C, and a vector in one of the bases, find the representation of the vector in the other basis. Verify that both vectors are equivalent to the same vector in the standard basis.

i. $\mathscr{B}=\left\{\begin{bmatrix}1\\3\end{bmatrix},\begin{bmatrix}-2\\0\end{bmatrix}\right\}, \mathscr{C}=\left\{\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}2\\-7\end{bmatrix}\right\}, [\vec{x}]_B=\begin{bmatrix}-4\\5\end{bmatrix}_B$

ii. $\mathscr{B}=\left\{\begin{bmatrix}1\\3\end{bmatrix},\begin{bmatrix}-2\\0\end{bmatrix}\right\}, \mathscr{C}=\left\{\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}2\\-7\end{bmatrix}\right\}, [\vec{x}]_C=\begin{bmatrix}1\\-1\end{bmatrix}_C$

iii. $\mathscr{B}=\left\{\begin{bmatrix}1\\2\\4\end{bmatrix},\begin{bmatrix}2\\1\\3\end{bmatrix},\begin{bmatrix}2\\2\\1\end{bmatrix}\right\}, \mathscr{C}=\left\{\begin{bmatrix}1\\-1\\2\end{bmatrix},\begin{bmatrix}0\\1\\1\end{bmatrix},\begin{bmatrix}3\\4\\5\end{bmatrix}\right\}, [\vec{x}]_B=\begin{bmatrix}15\\6\\-8\end{bmatrix}_B$

iv. $\mathscr{B}=\left\{\begin{bmatrix}1\\2\\4\end{bmatrix},\begin{bmatrix}2\\1\\3\end{bmatrix},\begin{bmatrix}2\\2\\1\end{bmatrix}\right\}, \mathscr{C}=\left\{\begin{bmatrix}1\\-1\\2\end{bmatrix},\begin{bmatrix}0\\1\\1\end{bmatrix},\begin{bmatrix}3\\4\\5\end{bmatrix}\right\}, [\vec{x}]_C=\begin{bmatrix}-4\\10\\11\end{bmatrix}_C$

v. $\mathscr{B}=\left\{\begin{bmatrix}1\\-2\\2\\-1\end{bmatrix},\begin{bmatrix}5\\1\\2\\1\end{bmatrix},\begin{bmatrix}2\\2\\1\\0\end{bmatrix},\begin{bmatrix}0\\0\\3\\1\end{bmatrix}\right\}, \mathscr{C}=\left\{\begin{bmatrix}2\\2\\0\\3\end{bmatrix},\begin{bmatrix}1\\1\\-1\\1\end{bmatrix},\begin{bmatrix}2\\0\\1\\0\end{bmatrix},\begin{bmatrix}1\\0\\0\\4\end{bmatrix}\right\}, [\vec{x}]_B=\begin{bmatrix}5\\2\\-3\\10\end{bmatrix}_B$

vi. $\mathscr{B}=\left\{\begin{bmatrix}1\\-2\\2\\-1\end{bmatrix},\begin{bmatrix}5\\1\\2\\1\end{bmatrix},\begin{bmatrix}2\\2\\1\\0\end{bmatrix},\begin{bmatrix}0\\0\\3\\1\end{bmatrix}\right\}, \mathscr{C}=\left\{\begin{bmatrix}2\\2\\0\\3\end{bmatrix},\begin{bmatrix}1\\1\\-1\\1\end{bmatrix},\begin{bmatrix}2\\0\\1\\0\end{bmatrix},\begin{bmatrix}1\\0\\0\\4\end{bmatrix}\right\}, [\vec{x}]_C=\begin{bmatrix}24\\12\\9\\13\end{bmatrix}_C$

vii. $\mathscr{B}=\left\{\begin{bmatrix}1\\0\\-1\end{bmatrix},\begin{bmatrix}2\\1\\0\end{bmatrix},\begin{bmatrix}2\\3\\1\end{bmatrix}\right\}, \mathscr{C}=\left\{\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\end{bmatrix}\right\}, [\vec{x}]_C=\begin{bmatrix}2\\-1\\3\end{bmatrix}_C$

f. For each of the B bases below, represent the vectors in the coordinate system of the C basis.

i. $\mathscr{B}=\left\{\begin{bmatrix}1\\3\end{bmatrix},\begin{bmatrix}-2\\0\end{bmatrix}\right\},\mathscr{C}=\left\{\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}2\\-7\end{bmatrix}\right\}$

ii. $\mathscr{B}=\left\{\begin{bmatrix}1\\2\\4\end{bmatrix},\begin{bmatrix}2\\1\\3\end{bmatrix},\begin{bmatrix}2\\2\\1\end{bmatrix}\right\},\mathscr{C}=\left\{\begin{bmatrix}1\\-1\\2\end{bmatrix},\begin{bmatrix}0\\1\\1\end{bmatrix},\begin{bmatrix}3\\4\\5\end{bmatrix}\right\}$

iii. $\mathscr{B}=\left\{\begin{bmatrix}1\\-2\\2\\-1\end{bmatrix},\begin{bmatrix}5\\1\\2\\1\end{bmatrix},\begin{bmatrix}2\\2\\1\\0\end{bmatrix},\begin{bmatrix}0\\0\\3\\1\end{bmatrix}\right\},\mathscr{C}=\left\{\begin{bmatrix}2\\2\\0\\3\end{bmatrix},\begin{bmatrix}1\\1\\-1\\1\end{bmatrix},\begin{bmatrix}2\\0\\1\\0\end{bmatrix},\begin{bmatrix}1\\0\\0\\4\end{bmatrix}\right\}$

**4. Orthogonal bases.**

If we are in the special case of an orthogonal basis, one where are the vectors of the basis are perpendicular to each other (the dot product of any basis vector with any other basis vector is zero), then there is a faster method for finding the entries of the vector representation in that basis.

If the basis $\mathscr{B}$ is an orthogonal basis with vectors $\{\overrightarrow{b_i}\}$, then the representation of the vector $\vec{x}$ in the basis $\mathscr{B}$ has entries

$$c_i = \frac{\vec{x}\cdot\overrightarrow{b_i}}{\overrightarrow{b_i}\cdot\overrightarrow{b_i}}$$

**Example 5**. Consider the basis $\mathscr{B}$ given by $\mathscr{B}=\left\{\begin{bmatrix}2\\3\end{bmatrix},\begin{bmatrix}-3\\2\end{bmatrix}\right\}$. The dot product of the two vectors with each other is zero, so this is an orthogonal basis. Find the representation of the vector $\vec{x}=\begin{bmatrix}5\\-7\end{bmatrix}$ in this basis.

$$c_1 = \frac{\vec{x}\cdot\overrightarrow{b_1}}{\overrightarrow{b_1}\cdot\overrightarrow{b_1}} = \frac{10-21}{4+9} = -\frac{11}{13}$$
$$c_2 = \frac{\vec{x}\cdot\overrightarrow{b_2}}{\overrightarrow{b_2}\cdot\overrightarrow{b_2}} = \frac{-15-14}{9+4} = -\frac{29}{13}$$

So $[\vec{x}]_B = \begin{bmatrix}-\frac{11}{13}\\-\frac{29}{13}\end{bmatrix}_B$. We can do a quick check, by using one of the other methods. $P_B^{-1}[\vec{x}]_B = \begin{bmatrix}\frac{2}{13} & \frac{3}{13}\\-\frac{3}{13} & \frac{2}{13}\end{bmatrix}\begin{bmatrix}5\\-7\end{bmatrix} = \begin{bmatrix}-\frac{11}{13}\\-\frac{29}{13}\end{bmatrix}_B$.

**Practice Problems.**

g. For each of the bases below, verify that the basis is orthogonal. Then use the method outlined in Example 5 to find the representation of the given vector in the basis.

i. $\mathscr{B}=\left\{\begin{bmatrix}1\\-2\end{bmatrix},\begin{bmatrix}2\\1\end{bmatrix}\right\}, \vec{x}=\begin{bmatrix}3\\5\end{bmatrix}$

ii. $\mathscr{C}=\left\{\begin{bmatrix}1\\2\\4\end{bmatrix},\begin{bmatrix}2\\-1\\0\end{bmatrix},\begin{bmatrix}-4\\-8\\5\end{bmatrix}\right\}, \vec{x}=\begin{bmatrix}15\\1\\-12\end{bmatrix}$

iii. $\mathscr{D}=\left\{\begin{bmatrix}1\\-2\\2\\-1\end{bmatrix},\begin{bmatrix}0\\0\\1\\2\end{bmatrix},\begin{bmatrix}2\\1\\0\\0\end{bmatrix},\begin{bmatrix}1\\-2\\-2\\1\end{bmatrix}\right\}, \vec{x}=\begin{bmatrix}5\\8\\-6\\11\end{bmatrix}$

iv. $\mathscr{T}=\left\{\begin{bmatrix}\frac{1}{\sqrt{21}}\\\frac{2}{\sqrt{21}}\\\frac{4}{\sqrt{21}}\end{bmatrix},\begin{bmatrix}\frac{2}{\sqrt{5}}\\-\frac{1}{\sqrt{5}}\\0\end{bmatrix},\begin{bmatrix}-\frac{4}{\sqrt{105}}\\-\frac{8}{\sqrt{105}}\\\frac{5}{\sqrt{105}}\end{bmatrix}\right\}, \vec{x}=\begin{bmatrix}3\\-2\\-4\end{bmatrix}$