Math B1131, Final Exam, Fall 2012 Name KEY

Instructions: Show all work. Use exact answers unless specifically asked to round. Be sure to complete all parts of each problem.

1. Integrate. (10 points each). [Hint: You may need to do some algebra before integrating.]
   a. $\int 5 - 2w - 6w^2\, dw$

$$\boxed{5w - w^2 - 2w^3 + C}$$

   b. $\int \frac{3}{\sqrt{x}} - 12\sqrt[3]{x} + \frac{1}{4e^x} dx$ $= \int 3x^{-1/2} - 12x^{1/3} + \frac{1}{4}e^{-x}\, dx$

$$3 \cdot 2x^{1/2} - \cancel{12}^{3} \cdot \frac{3}{\cancel{4}} x^{4/3} - \frac{1}{4}e^{-x} + C$$

$$\boxed{6\sqrt{x} - 9\sqrt[3]{x^4} - \frac{1}{4e^x} + C}$$

   c. $\int \frac{z^4+10z}{2z^2} + \frac{e^x+e^{2x}}{e^x} dx$ $= \int \frac{1}{2}x^2 + 5\frac{1}{x} + 1 + e^x\, dx$

$\frac{x^4+10x}{2x^2}$

$$\boxed{\frac{1}{6}x^3 + 5\ln x + x + e^x + C}$$

   d. $\int 4x^4(27 + x^5)^{1/3} dx$

$u = 27 + x^5$

$du = 5x^4\, dx$

$\frac{1}{5} du = x^4\, dx$

$$\int 4 \cdot \frac{1}{5} u^{1/3}\, du$$

$$\frac{\cancel{4}}{5} \cdot \frac{3}{\cancel{4}} u^{4/3} = \frac{3}{5} u^{4/3} \Rightarrow \boxed{\frac{3}{5}(27 + x^5)^{4/3} + C}$$

e. $\int \frac{1}{\sqrt{x-5}}dx$ $\quad u = x-5$

$du = dx$

$\int u^{-1/2}\,du$

$2u^{1/2} \Rightarrow \boxed{2\sqrt{x-5} + C}$

f. $\int \frac{16s-4}{3-2s+4s^2}ds$ $\quad u = 3-2s+4s^2$

$du = (-2+8s)ds = (8s-2)ds$

$\int \frac{2(8s-2)}{3-2s+4s^2}ds \Rightarrow \int \frac{2\,du}{u} = 2\ln u \Rightarrow \boxed{2\ln|3-2s+4s^2| + C}$

g. $\int \frac{8}{(x+3)\ln(x+3)}dx$ $\quad u = \ln(x+3)$

$du = \frac{1}{x+3}dx$

$\Rightarrow 8\int \frac{1}{u}du = 8\ln u \Rightarrow \boxed{8\ln|\ln(x+3)| + C}$

h. $\int 4xe^{-2x}dx$

$u = 4x \quad dv = e^{-2x}$

$du = 4\,dx \quad v = -\frac{1}{2}e^{-2x}$

$4x(-\frac{1}{2}e^{-2x}) - \int 4(-\frac{1}{2}e^{-2x})dx = -2xe^{-2x} + 2\int e^{-2x}dx =$

$\boxed{-2xe^{-2x} - e^{-2x} + C}$

or tabular:

	u	dv
+	$4x$	e^{-2x}
−	4	$-\frac{1}{2}e^{-2x}$
	0	$\frac{1}{4}e^{-2x}$

$= 4x(-\frac{1}{2}e^{-2x}) - 4(\frac{1}{4})e^{-2x} + C$

$= \boxed{-2xe^{-2x} - e^{-2x} + C}$

2. Integrate using the Fundamental Theorem of Calculus. Give exact answers. (15 points each)

a. $\int_1^3 3t^{-3}dt$

$$\frac{3t^{-2}}{-2}\Bigg|_1^3 = -\frac{3}{2}\left[\frac{1}{t^2}\right]_1^3 = -\frac{3}{2}\left[\frac{1}{9}-1\right] = -\frac{3}{2}\left[-\frac{8}{9}\right] = \boxed{\frac{4}{3}}$$

b. $\int_0^1 \frac{e^x-e^{-x}}{2}dx$

$$= \frac{1}{2}\int e^x - e^{-x}\,dx$$

$$= \frac{1}{2}\left(e^x + e^{-x}\right)\Bigg|_0^1 =$$

$$\frac{1}{2}\left[\left(e+e^{-1}\right) - (1+1)\right] = \boxed{\frac{1}{2}\left[e + 1/e - 2\right]}$$

c. $\int_2^{e+1} \frac{1}{x-1}dx$

$$\ln|x-1|\,\Bigg|_2^{e+1} = \ln|e+1-1| - \ln|2-1| =$$

$$\ln|e| - \underbrace{\ln 1}_{\to 0} = \boxed{1}$$

3. Use Simpson's Rule to approximate the value of the integral $\int_0^2 \frac{x}{x+1}\,dx$ using n=4. Round your answers to 4 decimal places. (15 points)

$\Delta x = h = \frac{2}{4} = \frac{1}{2}$

$0, \frac{1}{2}, 1, \frac{3}{2}, 2$

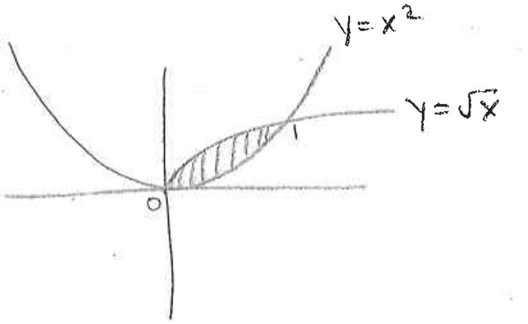
$\frac{1/2}{3}\left[f(0) + 4f(1/2) + 2f(1) + 4f(3/2) + f(2)\right] =$

$\frac{1}{6}\left[0 + 4\left(\frac{1/2}{3/2}\right) + 2\left(\frac{1}{2}\right) + 4\left(\frac{3/2}{5/2}\right) + \frac{2}{3}\right] =$

$\frac{1}{6}\left[0 + 4(1/3) + 1 + 4(3/5) + \frac{2}{3}\right] = \frac{1}{6}[5.4] = \boxed{.9}$

[Compare to true value of .9013877113...]

4. Find the area bounded by the graphs of $y = \sqrt{x}$ and $y = x^2$. Sketch the graph of the region. (20 points)



$(\sqrt{x})^2 = (x^2)^2$

$x = x^4$

$x^4 - x = 0$

$x(x^3 - 1) = 0$

$x = 0 \quad x^3 = 1 \Rightarrow x = 1$

$$\int_0^1 \sqrt{x} - x^2\,dx = \frac{2}{3}x^{3/2} - \frac{1}{3}x^3\Big|_0^1 = \frac{2}{3} - \frac{1}{3} - 0 = \boxed{\frac{1}{3}}$$

$\sqrt{x} \downarrow x^{1/2}$

5. The demand equation for a product is $p = 22 - 0.8q$ and the supply equation for the same product is $p = 6 + 1.2q$. Find the consumers' and producers' surplus. Round your answers to the nearest penny. (30 points)

$$22 - .8q = 6 + 1.2q$$
$$-6 \quad +.8q \quad -6 \quad +.8q$$

$$\frac{16}{2} = \frac{2q}{2}$$

$$q = 8$$

$$22 - .8(8) = 15.6$$

q_0, p_0

$$\boxed{(8, 15.6)}$$

$$CS = \int_0^8 22 - .8q - 15.6 \, dq =$$

$$\int_0^8 6.4 - .8q \, dq =$$

$$6.4q - .4q^2 \Big|_0^{14} = \boxed{25.6}$$

$$PS = \int_0^8 15.6 - (6 + 1.2q) \, dq =$$

$$\int_0^{14} 9.6 - 1.2q \, dq =$$

$$9.6q - .6q^2 \Big|_0^{14} = \boxed{38.4}$$

6. Solve the differential equation for $y(x)$ given $\frac{dy}{dx} = -x^3 y$ for the initial conditions $y(0) = e$. (15 points)

$$\int \frac{dy}{y} = \int -x^3 \, dx$$

$$\ln y = -\frac{1}{4}x^4 + C$$

$$\ln e = -\frac{1}{4}(0)^4 + C$$

$$C = 1$$

$$\boxed{\ln y = -\frac{1}{4}x^4 + 1}$$

$$y = e^{-\frac{1}{4}x^4 + C} = e^{-\frac{1}{4}x^4} e^C = Ae^{-\frac{1}{4}x^4}$$

let $A = e^C$

$$y = Ae^{-\frac{1}{4}x^4}$$

$$e = Ae^0 \quad A = e$$

$$\boxed{y = e^{-\frac{1}{4}x + 1}}$$

7. Suppose that for a certain radioactive element the half-life of the element is 13 weeks. a) Find the equation for the amount of material left after a certain period of time. b) Determine how much material is left at the end of 25 weeks. c) Determine how much material is left after 1 year. You may assume that the sample started with 100 grams of the material. Round your answers to two decimal places. (30 points)

$k = \frac{-\ln 2}{13}$

a) $A = A_0 e^{(-\ln 2/13)t}$

b) $A = 100 e^{-\frac{\ln 2}{13}(25)} = 26.37$ grams

c) $A = 100 e^{-\frac{\ln 2}{13}(52)} = 6.25$ grams

8. Find the indicated partial derivatives for the function $f(x, y, z) = -4x^2y^3z^2 + ye^{xyz}$. (10 points each)

a. f_x

$\boxed{-8xy^3z^2 + y^2ze^{xyz}}$

b. f_y $\boxed{-12x^2y^2z^2 + e^{xyz} + xyze^{xyz}}$

c. f_z

$\boxed{-8x^2y^3z + xy^2e^{xyz}}$

d. f_{xx} $\boxed{-8y^3z^2 + y^3z^2e^{xyz}}$

e. f_{zy} $\boxed{-24x^2y^2z + 2xye^{xyz} + x^2y^2ze^{xyz}}$

9. Use the chain rule to find $\frac{\partial z}{\partial t}$ for the equation $z = (x^2 + xy^2)^3, x = r^2 - t, y = 2r - 3s + 8t^3$.
(15 points)

$\frac{\partial z}{\partial x} = 3(x^2+xy^2)^2(2x+y^2)$

$\frac{\partial z}{\partial y} = 3(x^2+xy^2)^2(2xy)$

$\frac{\partial x}{\partial t} = -1$

$\frac{\partial y}{\partial t} = 24t^2$

$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} =$

$$\boxed{\begin{aligned}&3\left[(r^2-t)^2 + (r^2-t)(2r-3s+8t^3)^2\right]^2\left[2(r^2-t) + (2r-3s+8t^3)^2\right](-1)\\ &+ 3\left[(r^2-t)^2+(r^2-t)(2r-3s+8t^3)^2\right]^2\left[2(r^2-t)(2r-3s+8t^3)\right](24t^2)\end{aligned}}$$

10. Find all the critical points for the function $f(x, y) = x^2 + 3xy + y^2 - 9x - 11y + 3$. For each critical point, determine if the point is a maximum, minimum, saddle point or if it cannot be determined. (15 points)

$f_x = 2x + 3y - 9 = 0 \quad \times 2 \Rightarrow \quad 4x + \cancel{6y} - 18 = 0$

$f_y = 3x + 2y - 11 = 0 \quad \times -3 \quad\quad -9x - \cancel{6y} + 33 = 0$

$-5x + 15 = 0$

$5x = 15 = 0$

$\boxed{x = 3}$

$2(3) + 3y - 9 = 0$

$3y - 3 = 0$

$3y = 3$

$\boxed{y = 1}$

$(3, 1)$

$f_{xx} = 2 \quad f_{xy} = 3 \quad D: 2(2) - 3^2 = 4 - 9 = -5 < 0$

$f_{yy} = 2$

$\boxed{(3,1) \text{ is a saddle point}}$

11. Use Lagrange Multipliers to find all the critical points of the function $f(x, y, z) = x^2 + y^2 + z^2$ subject to the constraint $2x + y - z = 9$. (15 points)

$F(x, y, z, \lambda) = x^2 + y^2 + z^2 - 2\lambda x - \lambda y + \lambda z + 9\lambda$

$F_x = 2x - 2\lambda = 0 \quad \lambda = x$

$x = 2y$

$F_y = 2y - \lambda = 0 \quad \lambda = 2y$

$F_z = 2z + \lambda = 0 \quad \lambda = -2z$

$2y = -2z \quad -y = +z$

$(3, 3/2, -3/2, 27/2)$

$2(2y) + y - (-y) = 9$

$4y + y + y = 9$

$6y = 9$

$y = \frac{3}{2}$

$9 + (3/2)^2 + (-3/2)^2 =$

$9 + \frac{9}{4} + \frac{9}{4} = 27/2$

$x = 3 \quad z = -3/2$