Math 2153, Exam #3, Fall 2012 Name KEY

Instructions: Show all work. Use exact answers unless specifically asked to round or complete a problem numerically.

1. Integrate. (5 points each)

a. $\int x\,y^2 \cosh(z)\, dy$

$$\boxed{\frac{1}{3}xy^3\cosh(z) + C(x,z)}$$

b. $\int_{-4}^{4}\int_{0}^{x^2}\sqrt{64-x^3}\,dydx$

$$\int_{-4}^{4} y\sqrt{64-x^3}\Big|_0^{x^2} = \int_{-4}^{4} x^2\sqrt{64-x^3}\,dx$$

$u = 64-x^3$

$du = -3x^2\,dx$

$-\frac{1}{3}du = x^2\,dx$

$$\Rightarrow \int -\frac{1}{3}u^{1/2}\,du = -\frac{1}{3}\cdot\frac{2}{3}u^{3/2} \Rightarrow -\frac{2}{9}(64-x^3)^{3/2}\Big|_{-4}^{4}$$

$$-\frac{2}{9}\left[0-\underbrace{(128)^{3/2}}_{(8\sqrt{2})^3}\right] = \boxed{\frac{2048\sqrt{2}}{9}}$$

c. $\int_{1}^{\infty}\int_{0}^{1/x} y\,dydx$

$$\int_1^{\infty}\frac{1}{2}y^2\Big|_0^{1/x} = \int_1^{\infty}\frac{1}{2x^2}\,dx = \frac{1}{2}\lim_{b\to\infty}\int_1^b x^{-2}\,dx =$$

$$\frac{1}{2}\lim_{b\to\infty} -\frac{1}{x}\Big|_1^b = \frac{1}{2}\lim_{b\to\infty}\left[\cancelto{0}{-\frac{1}{b}}+1\right] = \boxed{\frac{1}{2}}$$

d. $\int_{1}^{4}\int_{1}^{e^2}\int_{0}^{\frac{1}{xz}}\ln(z)\,dydzdx$

$$\int_1^4\int_1^{e^2} y\ln z\Big|_0^{1/xz}\,dz\,dx = \int_1^4\int_1^{e^2}\frac{1}{x}\cdot\frac{\ln z}{z}\,dz\,dx$$

$u = \ln z$

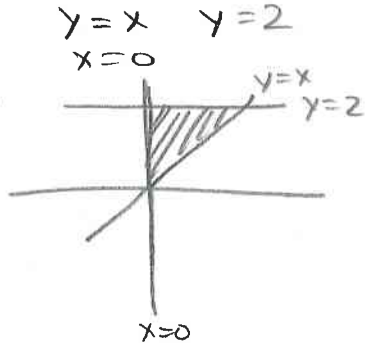
$du = \frac{1}{z}dz$

$$\Rightarrow \int u\,du \Rightarrow \frac{1}{2}u^2 \Rightarrow \int_1^4 \frac{1}{2}\frac{1}{x}(\ln z)^2\Big|_1^{e^2}dx =$$

$$\int_1^4 \frac{1}{2}\frac{1}{x}\left[(\ln e^2)^2-(\ln(1))^2\right]dx = \int_1^4\frac{1}{2}\frac{1}{x}\cdot[4-0]\,dx = \int_1^4\frac{2}{x}\,dx =$$

$$2\ln x\Big|_1^4 = 2\ln 4 - \cancelto{0}{2\ln 1} = 2\ln 4$$

2. Sketch the region defined by $\int_0^2 \int_x^2 e^{-y^2} dydx$, then change the order of integration and complete the integration. (15 points)



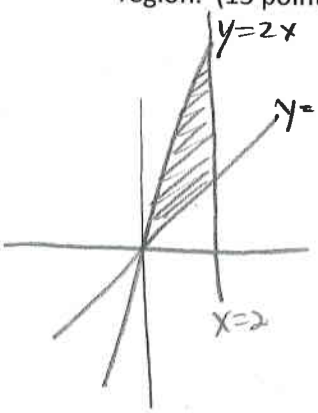
$$\int_0^2 \int_0^y e^{-y^2}\, dx\, dy = \int_0^2 x e^{-y^2} \Big|_0^y dy =$$

$$\int_0^2 y e^{-y^2} dy \qquad \begin{aligned} u &= -y^2 \\ du &= -2y\,dy \\ -\tfrac{1}{2} du &= y\,dy \end{aligned}$$

$$\Rightarrow -\frac{1}{2}\int e^u du = -\frac{1}{2} e^u \Rightarrow -\frac{1}{2} e^{-y^2}\Big|_0^2 = -\frac{1}{2}\left[e^{-4} - e^0\right] =$$

$$\boxed{\frac{1}{2}\left[1 - \frac{1}{e^4}\right]}$$

3. Find the area of the region bounded by $y = x, y = 2x, x = 2$ using a double integral. Sketch the region. (15 points)



$$\int_0^2 \int_x^{2x} 1\, dy\, dx$$

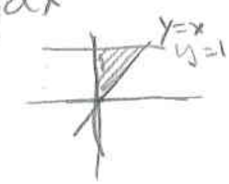
$$\int_0^2 y \Big|_x^{2x} dx = \int_0^2 2x - x\, dx$$

$$\int_0^2 x\, dx = \frac{1}{2} x^2 \Big|_0^2 =$$

$$\frac{1}{2}[4 - 0] = \boxed{2}$$

4. Find the volume of the solid bounded by $z = 1 - xy, y = x, y = 1$, in the first octant. (15 points)

Double: $\int_0^1\int_x^1 1-xy \, dy\, dx$ Triple: $\int_0^1\int_x^1\int_0^{1-xy} 1 \, dz\, dy\, dx$



$$\int_0^1 y - \frac{1}{2}xy^2\Big|_x^1 dx = \int_0^1 1 - \frac{1}{2}x - x + \frac{1}{2}x^3 \, dx =$$

$$\int_0^1 1 - \frac{3}{2}x + \frac{1}{2}x^3 \, dx = x - \frac{3}{4}x^2 + \frac{1}{8}x^4\Big|_0^1 =$$

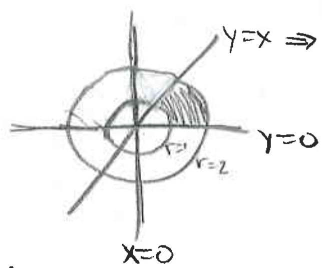
$$1 - \frac{3}{4} + \frac{1}{8} = \frac{1}{4} + \frac{1}{8} = \boxed{\frac{3}{8}}$$

5. Find the volume of the region bounded by $f(x,y) = \arctan\left(\frac{y}{x}\right), x^2 + y^2 \geq 1, x^2 + y^2 \leq 4, 0 \leq y \leq x$. Sketch the region in the plane. (20 points)

$\arctan(y/x) = \theta$

$y \geq 0,\ x \geq 0$

$y \leq x$



$$\int_0^{\pi/4}\int_1^2 \theta\, r\, dr\, d\theta =$$

$$\int_0^{\pi/4} \theta \frac{1}{2}r^2\Big|_1^2 d\theta = \int_0^{\pi/4} \theta\left(\frac{1}{2}(4) - \frac{1}{2}(1)\right) d\theta$$

$$= \int_0^{\pi/4} \frac{3}{2}\theta \, d\theta = \frac{3}{4}\theta^2\Big|_0^{\pi/4} = \frac{3}{4}\left[\left(\frac{\pi}{4}\right)^2\right] =$$

$$\frac{3\pi^2}{64}$$

6. Set up but the integrals for finding the center of mass for the three-dimensional region $z = 9 - x^2, y = -x + 2$, first octant with density $\rho(x,y,z) = kxz^2(y+2)^2$. Do not integrate. (15 points)

$x=0, y=0$
$z=0$

$$M = \int_0^2 \int_0^{-x+2} \int_0^{9-x^2} kxz^2(y+2)^2 \, dz\,dy\,dx$$

$$M_{yz} = \int_0^2 \int_0^{-x+2} \int_0^{9-x^2} kx^2z^2(y+2)^2 \, dz\,dy\,dx$$

$$M_{xz} = \int_0^2 \int_0^{-x+2} \int_0^{9-x^2} kxyz^2(y+2)^2 \, dz\,dy\,dx$$

$$M_{xy} = \int_0^2 \int_0^{-x+2} \int_0^{9-x^2} kxz^3(y+2)^2 \, dz\,dy\,dx$$

$$(\bar{x}, \bar{y}, \bar{z}) = \left(\frac{M_{yz}}{M}, \frac{M_{xz}}{M}, \frac{M_{xy}}{M}\right)$$

7. Find the integral for the surface area of $f(x,y) = 9 + x^2 - y^2$ over the region $R: \{(x,y) | x^2 + y^2 < 16\}$. Do complete the integration. (20 points)

$f_x = 2x$
$f_y = -2y$

$$\sqrt{1 + (2x)^2 + (-2y)^2} = \sqrt{1+4x^2+4y^2} = \sqrt{1+4r^2}$$

$$\int_0^{2\pi}\int_0^4 \sqrt{1+4r^2}\, r\,dr\,d\theta =$$

$u = 1+4r^2$
$du = 8r\,dr$
$\frac{1}{8}du = r\,dr$

$$\Rightarrow \int u^{1/2} \cdot \frac{1}{8} du = \frac{2}{3}\cdot\frac{1}{8} u^{3/2} \Rightarrow \int_0^{2\pi} \frac{1}{12}(1+4r^2)^{3/2}\Big|_0^4 d\theta$$

$$\int_0^{2\pi} \frac{1}{12}[65^{3/2} - 1]\,d\theta = \boxed{\frac{\pi}{6}[65^{3/2} - 1]}$$

8. Find ^the volume of^ the solid inside the sphere $x^2+y^2+z^2=16$ and outside $z=\sqrt{x^2+y^2}$. (15 points)

$\rho=4 \qquad \varphi=\pi/4$

$$\int_0^{2\pi}\int_{\pi/4}^{\pi}\int_0^4 \rho^2\sin\varphi\, d\rho\, d\varphi\, d\theta$$

$$\int_0^{2\pi}\int_{\pi/4}^{\pi}\frac{1}{3}\rho^3\Big|_0^4 \sin\varphi\, d\varphi\, d\theta = \int_0^{2\pi}\int_{\pi/4}^{\pi}\frac{64}{3}\sin\varphi\, d\varphi\, d\theta =$$

$$\int_0^{2\pi}\frac{64}{3}(-\cos\varphi)\Big|_{\pi/4}^{\pi} d\theta = \int_0^{2\pi}\frac{64}{3}\left[1+\frac{\sqrt{2}}{2}\right] d\theta = \boxed{\frac{64\pi(2+\sqrt{2})}{3}}$$

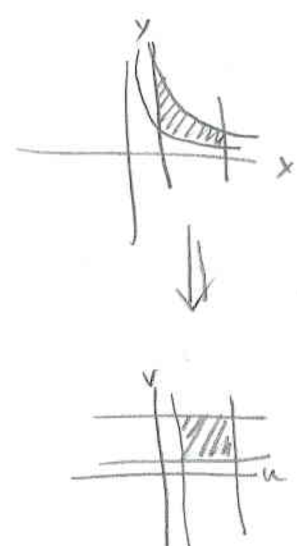
9. Use change of variables to calculate the volume of the solid bounded by $f(x,y)=\frac{xy}{1+x^2y^2}$, $xy=1$, $xy=4$, $x=1$, $x=4$. (15 points)

$u=xy \qquad x=v$

$u=v\cdot y \Rightarrow y=\frac{u}{v}$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ \frac{1}{v} & -\frac{u}{v^2} \end{vmatrix} = 0-\frac{1}{v} = -\frac{1}{v}$$

$f(u,v)=\frac{u}{1+u^2} \qquad 1\le u\le 4,\ 1\le v\le 4$



$$\int_1^4\int_1^4 -\frac{u}{1+u^2}\cdot\frac{1}{v}\, du\, dv$$

$w=1+u^2$

$dw=2u\,du$

$\frac{1}{2}dw=u\,du$

$\Rightarrow \int\frac{1}{2}\frac{1}{w}dw = \frac{1}{2}\ln|w| \Rightarrow$

$$\int_1^4 -\frac{1}{2}\ln|1+u^2|\Big|_1^4 \frac{1}{v}dv = \int_1^4 -\frac{1}{2}[\ln 17-\ln 2]\cdot\frac{1}{v}dv$$

$$-\frac{1}{2}\ln(17/2)\ln v\Big|_1^4 = -\frac{1}{2}\ln(17/2)(\ln 4-\cancel{\ln 1}^{0}) = -\frac{1}{2}\ln(17/2)\ln 4$$

neglect the negative $\Rightarrow \boxed{\frac{1}{2}\ln(17/2)\ln 4}$

10. Find the Jacobian for the transformation to spherical coordinates. (15 points)

$x = \rho \cos\theta \sin\varphi$
$y = \rho \sin\theta \sin\varphi$
$z = \rho \cos\varphi$

$$\frac{\partial(x,y,z)}{\partial(\rho,\varphi,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \varphi} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \varphi} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \varphi} & \frac{\partial z}{\partial \theta} \end{vmatrix} =$$

$$\begin{vmatrix} \cos\theta \sin\varphi & \rho \cos\theta \cos\varphi & -\rho \sin\theta \sin\varphi \\ \sin\theta \sin\varphi & \rho \sin\theta \cos\varphi & \rho \cos\theta \sin\varphi \\ \cos\varphi & -\rho \sin\varphi & 0 \end{vmatrix} =$$

$$\cos\theta \sin\varphi \, (\underbrace{\cancel{\rho \sin\theta \cos\varphi (0)}}_{\to 0} + (+\rho \sin\varphi \cdot \rho \cos\theta \sin\varphi))$$

$$- \rho \cos\theta \cos\varphi \, (\underbrace{\cancel{\sin\theta \sin\varphi (0)}}_{\to 0} - \cos\varphi \cdot \rho \cos\theta \sin\varphi)$$

$$+ (-\rho \sin\theta \sin\varphi)(-\rho \sin\varphi \cdot \sin\theta \sin\varphi - \rho \sin\theta \cos\varphi \cdot \cos\varphi) =$$

$$\rho^2 \sin\varphi \, (\cos^2\theta \sin^2\varphi) + \rho^2 \sin\varphi \, (\cos^2\theta \cos^2\varphi) + \rho^2 \sin\varphi \, (\sin^2\theta \sin^2\varphi + \sin^2\theta \cos^2\varphi)$$

$$= \rho^2 \sin\varphi \left[\cos^2\theta \sin^2\varphi + \cos^2\theta \cos^2\varphi + \sin^2\theta \sin^2\varphi + \sin^2\theta \cos^2\varphi \right] =$$

$$\rho^2 \sin\varphi \left[\cos^2\theta \, (\underbrace{\cancel{\sin^2\varphi + \cos^2\varphi}}_{=1}) + \sin^2\theta \, (\underbrace{\cancel{\sin^2\varphi + \cos^2\varphi}}_{=1}) \right] =$$

$$\rho^2 \sin\varphi \, [\underbrace{\cancel{\cos^2\theta + \sin^2\theta}}_{=1}] \quad = \boxed{\rho^2 \sin\varphi}$$