Math 2153, Exam #2, Fall 2012 Name KEY

Instructions: Show all work. Use exact answers unless specifically asked to round. Be sure to complete all parts of each problem.

1. Draw six level curves for the function $f(x,y) = \frac{x}{x^2+y^2}$. Use both positive and negative values for z. (10 points)

$z = \frac{x}{x^2+y^2}$ $\quad z=0 \Rightarrow x=0$

$x^2+y^2 = \frac{x}{z}$

$\sqrt{y^2} = \pm\sqrt{\frac{x}{z} - x^2}$

$z=1 \quad y = \pm\sqrt{x-x^2}$

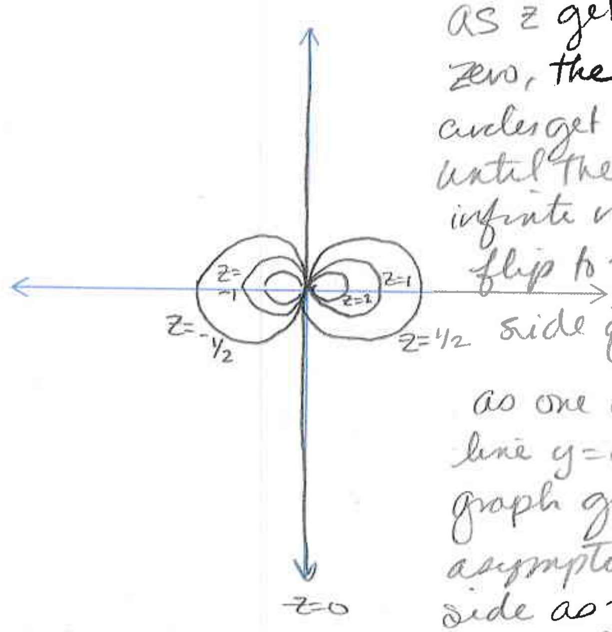
$z=2 \quad y = \pm\sqrt{\frac{1}{2}x - x^2}$

$z=\frac{1}{2} \quad y = \pm\sqrt{2x-x^2}$

$z=-\frac{1}{2} \quad y = \pm\sqrt{-2x-x^2}$

$z=-1 \quad y = \pm\sqrt{-x-x^2}$

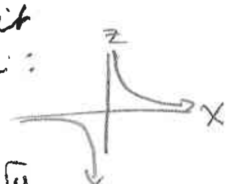
$z=-2 \quad y = \pm\sqrt{-\frac{1}{2}x - x^2}$



as z gets closer to zero, the cross-sectional circles get larger until they have infinite radius, then flip to the other side of the graph

as one approaches line y=0 the graph goes to an asymptote on either side as the cross-sectional circles go to zero radius in the limit

in profile:



2. Find the limits if they exist, or prove that they do not exist. (5 points each)

a. $\lim_{(x,y)\to(0,0)} \frac{x-y}{\sqrt{x}-\sqrt{y}} \cdot \frac{\sqrt{x}+\sqrt{y}}{\sqrt{x}+\sqrt{y}} = \lim \frac{\cancel{(x-y)}(\sqrt{x}+\sqrt{y})}{\cancel{x-y}} = \lim_{(x,y)\to(0,0)} \sqrt{x}+\sqrt{y}$

$\boxed{= 0}$

b. $\lim_{(x,y)\to(0,0)} \frac{x^3 y}{x^6+y^2}$ $\quad$ path $x=0$ $\quad \lim_{y\to 0} \frac{0}{y^2} = 0$ $\quad \boxed{\text{DNE}}$

$x^6 = y^2$

$y = x^3$ path $y = kx^3$

$\lim_{x\to 0} \frac{x^3 \cdot kx^3}{x^6 + k^2x^6} = \lim_{x\to 0} \frac{k\cancel{x^6}}{\cancel{x^6}(1+k^2)} = \lim_{x\to 0} \frac{k}{1+k^2} = \frac{k}{1+k^2}$

$\neq 0$ if $k \neq 0$

c. $\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{3y^4+x^4}$ in polar $\lim_{r\to 0} \frac{r^2\cos^2\theta\, r^2\sin^2\theta}{3r^4\sin^4\theta + r^4\cos^4\theta} =$

$\lim_{r\to 0} \frac{\cancel{r^4}(\cos^2\theta\sin^2\theta)}{\cancel{r^4}(3\sin^4\theta+\cos^4\theta)} = \frac{\cos^2\theta\sin^2\theta}{3\sin^4\theta+\cos^4\theta}$

depends on θ so $\boxed{\text{DNE}}$ (or use path $y=kx$)

d. $\lim_{(x,y,z)\to(0,0,0)} \frac{xy+yz+xz}{x^2+y^2+z^2}$ in spherical

$\lim_{\rho\to 0} \frac{\cancel{\rho^2}\sin^2\varphi\cos\theta\sin\theta + \cancel{\rho^2}\sin\varphi\sin\theta\cos\varphi + \cancel{\rho^2}\sin\varphi\cos\theta\cos\varphi}{\cancel{\rho^2}}$

depends on φ and θ so $\boxed{\text{DNE}}$

3. Find the indicated partial derivatives for the function $f(x,y,z) = e^{-x}\sin(yz)$. (5 points each)

a. $\frac{\partial f}{\partial x} = \boxed{-e^{-x}\sin yz}$

$f_y = ze^{-x}\cos(yz)$

b. $\frac{\partial f}{\partial z} = \boxed{ye^{-x}\cos(yz)}$

c. $f_{yz} = \boxed{e^{-x}\cos(yz) - yze^{-x}\sin(yz)}$

$= e^{-x}(\cos(yz) - yz\sin(yz))$

d. f_{yzx}

$= \boxed{-e^{-x}(\cos(yz) - yz\sin(yz))}$

4. Find the total differential of the function $f(x,y,z) = e^y \cos(x) + z^2$. Evaluate it at the point $(\pi, 0, 2)$, and use it to approximate the value of the function at $\left(\frac{7\pi}{6}, -0.1, \frac{9}{4}\right)$. (10 points)

$$dw = f_x dx + f_y dy + f_z dz$$
$$(-\sin(x)e^y)dx + (e^y \cos(x))dy + (2z)dz$$
$$= 0\,dx + (-1)\,dy + 4\,dz$$

$$f(\pi, 0, 2) = 1(-1) + 4 = -1+4 = 3$$

$$f(7\pi/6, -0.1, 9/4) \approx 3 + [0(\pi/6) - 1(-.1) + 4(\tfrac{1}{4})] =$$
$$3 + .1 + 1 = \boxed{4.1}$$

5. Find $\frac{dw}{dt}$ for the following $w = xy^2 + x^2z + yz^2, x = t^2, y = \arccos(t),\ z = e^{-2t}$ using the chain rule. Be sure your final answers contain only t. (10 points)

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

$$\frac{dx}{dt} = 2t$$

$$\frac{\partial w}{\partial x} = y^2 + 2xz = \arccos^2 t + 2t^2 e^{-2t}$$

$$\frac{dy}{dt} = \frac{-1}{\sqrt{1-t^2}}$$

$$\frac{\partial w}{\partial y} = 2xy + z^2 = 2t^2 \arccos t + e^{-4t}$$

$$\frac{dz}{dt} = -2e^{-2t}$$

$$\frac{\partial w}{\partial z} = x^2 + 2yz = t^4 + 2e^{-2t}\arccos t$$

$$\boxed{\frac{dw}{dt} = (\arccos^2 t + 2t^2 e^{-2t})2t - \frac{(2t^2 \arccos t + e^{-4t})}{\sqrt{1-t^2}} - 2e^{-2t}(t^4 + 2e^{-2t}\arccos t)}$$

6. Find the implicit partial derivative $\frac{\partial w}{\partial y}$ for the function $x^2 + y^2 - yzw + 3zw^4 - \cosh(y) = 0$ by **both** means (the long way **and** by means of the short-cut formulas). Verify that both produce the same results. (15 points)

long way:

$$2y - zw - yzw_y + 12zw^3 w_y - \sinh(y) = 0$$

$$2y - zw - \sinh(y) = yzw_y - 12zw^3 w_y$$

$$\frac{2y - zw - \sinh(y)}{yz - 12zw^3} = \frac{\cancel{(yz - 12zw^3)} w_y}{\cancel{yz - 12zw^3}}$$

$$\boxed{\frac{\partial w}{\partial y} = w_y = \frac{2y - zw - \sinh(y)}{yz - 12zw^3}}$$

Short way $F(x,y,z,w) = x^2 + y^2 - yzw + 3zw^4 - \cosh(y)$:

$$\boxed{\frac{\partial w}{\partial y} = -\frac{F_y}{F_w} = -\frac{2y - zw - \sinh(y)}{-yz + 12zw^3}}$$ checks ✓

7. Find the directional derivative for the function $f(x, y, z) = x\tan(y - z)$ at the point (4,3,1) in the direction of $\vec{v} = \hat{\imath} - 2\hat{\jmath} + 3\hat{k}$. Then state the maximum value of the directional derivative at that point. (10 points)

$$\nabla f = \langle \tan(y-z), x\sec^2(y-z), -x\sec^2(y-z) \rangle$$

$$\nabla f(4,3,1) = \langle \tan(2), 4\sec^2(2), -4\sec^2(2) \rangle$$

$$\|\vec{v}\| = \sqrt{1 + 2^2 + 3^2} = \sqrt{1+4+9} = \sqrt{14}$$

$$\hat{u} = \frac{1}{\sqrt{14}}\hat{\imath} - \frac{2}{\sqrt{14}}\hat{\jmath} + \frac{3}{\sqrt{14}}\hat{k}$$

max value $= \sqrt{\tan^2 2 + 16\sec^4 2 + 16\sec^4 2}$

$= \boxed{\sqrt{\tan^2 2 + 32\sec^4 2}}$

≈ 32.78

$$\nabla f \cdot \hat{u} = \langle \tan(2), 4\sec^2(2), -4\sec^2(2) \rangle \cdot \langle \frac{1}{\sqrt{14}}, \frac{-2}{\sqrt{14}}, \frac{3}{\sqrt{14}} \rangle =$$

$$\frac{\tan 2}{\sqrt{14}} - \frac{8\sec^2(2)}{\sqrt{14}} - \frac{12\sec^2 2}{\sqrt{14}} = \boxed{\frac{\tan 2 - 20\sec^2 2}{\sqrt{14}}} \approx -31.45$$

8. Find an equation of the tangent plane to the surface at $xy^2 + 3x - z^2 = 4$ at the point (2,1,-2). Then state the equation of the normal line at the same point. (10 points)

$F(x,y,z) = xy^2 + 3x - z^2 - 4$

$\nabla F = \vec{n} = \langle y^2+3,\ 2xy,\ -2z \rangle$

$\nabla F(2,1,-2) = \langle 1+3,\ 2(2)(1),\ -2(-2) \rangle = \langle 4, 4, 4 \rangle$

tangent plane: $\boxed{4(x-2) + 4(y-1) + 4(z+2) = 0}$

normal line: symmetric $\boxed{\dfrac{x-2}{4} = \dfrac{y-1}{4} = \dfrac{z+2}{4}}$

vector-valued $(4t+2)\hat{i} + (4t+1)\hat{j} + (4t-2)\hat{k} = \vec{r}(t)$

9. Find the critical points for the function $f(x,y) = x^3 + 6xy + 10y^2 + \boxed{2y} + 4$, and characterize each point as either a maximum, minimum, saddle point, or cannot be determined. (10 points)

4y

$f_x = 3x^2 + 6y = 0 \Rightarrow \dfrac{-6y}{3} = \dfrac{3x^2}{3} \Rightarrow -2y = x^2 \Rightarrow y = -\frac{1}{2}x^2$

$f_y = 6x + 20y + 4 = 0$

$6x - 10(-2y) + 4 = 0$

$6x - 10x^2 + 4 = 0$

$10x^2 - 6x - 4 = 0$

$5x^2 - 3x - 2 = 0$

$x = \dfrac{+3 \pm \sqrt{9 - 4(5)(-2)}}{10}$

$\sqrt{9+40} = \sqrt{49} = 7$

$x = \dfrac{3 \pm 7}{10} = \dfrac{10}{10}, \dfrac{-4}{10} = 1, -\dfrac{2}{5}$

$x = 1 \quad y = -\frac{1}{2} \qquad (1, -\frac{1}{2})$

$x = -\frac{2}{5} \quad y = -\frac{2}{25} \qquad (-\frac{2}{5}, -\frac{2}{25})$

$f_{xx} = 6x$

$f_{yy} = 20$

$f_{xy} = 6$

$D(1, -\frac{1}{2}) = 6(20) - 6^2 = 120 - 36 > 0$

$f_{xx}(1, -\frac{1}{2}) = 6 > 0 \ \cup$

$\boxed{(1, -\tfrac{1}{2}) \text{ is a minimum}}$

$D(-\frac{2}{5}, -\frac{2}{25}) = (-\frac{12}{5})(20) - 6^2 < 0$

$\boxed{(-\tfrac{2}{5}, -\tfrac{2}{25}) \text{ is a saddle point}}$

10. Find the absolute extrema for the function $f(x,y) = x^2 - 4xy + 5$ over the region $R: \{(x,y) | 0 \le x \le 4, 0 \le y \le \sqrt{x}\}$. Sketch the region. (15 points)

① critical points

$f_x = 2x - 4y = 0 \Rightarrow 2x = 4y \Rightarrow x = 2y$

$f_y = -4x = 0 \quad x = 0 \Rightarrow y = 0 \quad (0,0)$

② boundaries (x=0 is not a true boundary eq.)

a) $y = 0$

$f(x,0) = x^2 + 5$

$f'(x,0) = 2x \quad x = 0 \quad (0,0)$ (again)

b) $x = 4$

$f(4,y) = 16 - 16y + 5 = -16y + 21$

$f'(4,y) = -16 \neq 0$

c) $y = \sqrt{x}$

$f(x,\sqrt{x}) = x^2 - 4x^{3/2} + 5$

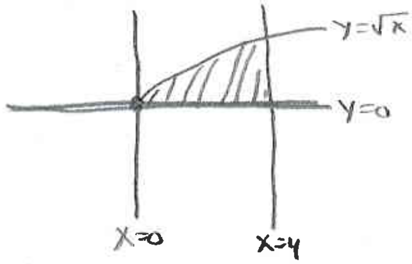
$f'(x,\sqrt{x}) = 2x - 6x^{1/2} = 0$

$x^{1/2}(2\sqrt{x} - 6) = 0$

$x = 0$ ∮ $2\sqrt{x} = 6 \Rightarrow \sqrt{x} = 3 \quad x = 9$

$x = 0 \Rightarrow (0,0)$ again

$x = 9$ outside region



③ Corners

(0,0)

(4,0)

(4,2)

pts to check

$f(0,0) = 5$

$f(4,0) = 16 - 0 + 5 = 21$

$f(4,2) = 16 - 32 + 5 = -11$

@ (4,0) absolute max of 21

@ (4,2) absolute min of −11

11. Minimize the function $f(x,y) = 2x + y$ subject to the constraint $xy = 32$ using Lagrange Multipliers. (10 points)

$F(x,y,\lambda) = 2x + y - \lambda(xy - 32)$

$F_x = 2 - \lambda y = 0 \Rightarrow 2 = \lambda y \Rightarrow \lambda = \frac{2}{y}$

$F_y = 1 - \lambda x = 0 \quad 1 = \lambda x \Rightarrow \lambda = \frac{1}{x}$

$\Rightarrow \frac{2}{y} = \frac{1}{x} \Rightarrow 2x = y$

into constraint

$x(2x) = 32$

$2x^2 = 32$

$x^2 = 16$

$x = \pm 4$

$\Rightarrow (4,8)$

$(-4,-8)$

$f(4,8) = 2(4) + 8 = 16$ Maximum

$f(-4,-8) = 2(-4) - 8 = -16$ Minimum