Teaching Epicycles & Fourier Series in the Classroom

Short Paper

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The math behind epicycles and Fourier series are at two vastly different levels, so I'm going to take the two topics apart first, and then we'll see if we can bring them together again at the end. Let's start by considering Fourier series. I've taught this material to students in college/university before in a course that included some partial differential equations. To teach the material directly, students need two semesters of calculus, at a minimum. But there are some elements of Fourier series that could be addressed earlier. For instance, Fourier series are used to model periodic functions, and before we can model our non-periodic functions, we have to make them periodic. We discuss piecewise functions and symmetry in algebra classes. We could discuss the math of periodicity before we look at naturally periodic functions like trig functions. One skill we have to pause to work on for Fourier series is extending a non-periodic function into a periodic one, and then choosing which type of symmetry (if any) to use to do the extension. For example, suppose I have the function $f(x) = x, 0 \leq x < 5$ (where I am using 5 to stand in for L) as shown here. This function is not periodic, and ideally, we'd like it to extend on the interval $[-L, L]$. But how do we do that?

There are generally three main options: 1) extend on $[-L, 0]$ and let the function there be zero; 2) extend with even symmetry, 3) extend with odd symmetry.

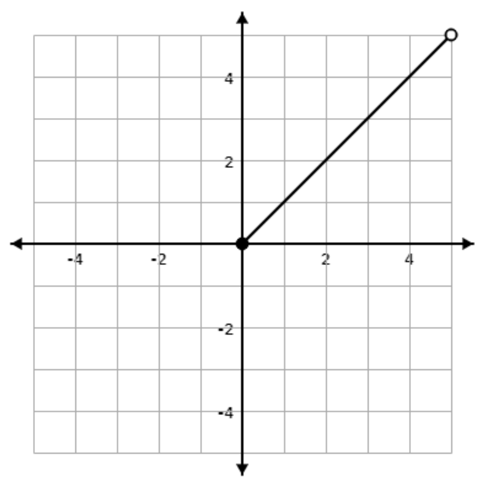
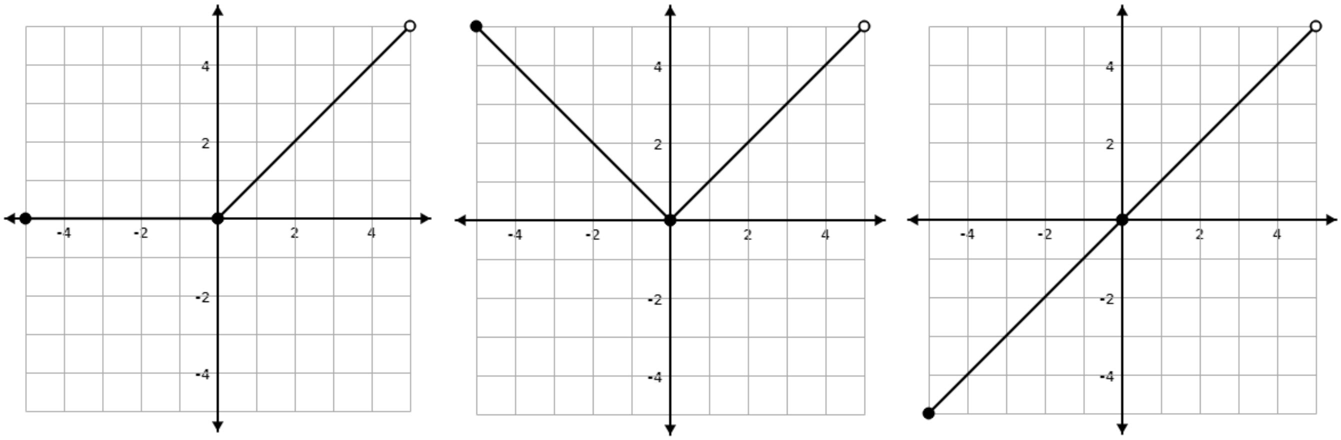


Figure 1. Graph of $f(x)$.

(GraphFree.com, 2021)



$$f(x) = \begin{cases} 0, -5 \le x < 0 \\ x, 0 \le x < 5 \end{cases} \qquad f(x) = \begin{cases} -x, & -5 \le x < 0 \\ x, & 0 \le x < 5 \end{cases} \qquad f(x) = \begin{cases} x, & -5 \le x < 0 \\ x, & 0 \le x < 5 \end{cases}$$

Figure 2. Extensions of the function on the interval [-5,5], produced by the author using GraphFree.

Once extended on $[-L, L]$, then the function can be extended periodically. These extensions have implications for Fourier series, since the first one will include both sine and cosines, the second would have only cosines, and the third one only sines. While the Fourier series themselves can't be developed at this point, technology may be able to create the model to see the results of the extensions, although that might involve waiting until after the introduction of trig functions.

Another natural place to introduce the idea of Fourier series is with dealing with complex sounds. Fourier transforms are often used to model speech or music. It's a lot easier to understand how different types of waveforms can be combined to produce another more complex waveform than it is how they can combine to make a straight line. Technology would have to do all the computations here, of course, but the concept of combining waves could be explored. This level of conceptual introduction to the idea could be done much earlier, even in a music class. I remember my music teacher putting sand on a drum and watching the different patterns form, and it wasn't until much later that I understood this was a multivariable PDE problem.

With respect to epicycles, they don't require calculus, so there are many places where these ideas could be discussed in depth at the high school math level. A fairly rigorous treatment could be done in a precalculus course, once trig functions and parametric equations are introduced. This is the point when we introduce polar graphs, and this would be a natural extension of these ideas. Extending out either the x or y coordinate to a horizontal graph could then connect the epicycles to the one-variable Fourier series (TivnanR, 2018). Certainly, they could be discussed in geometry from a geometric perspective, but as I recall from my own class, the topic was entirely a static discussion, so incorporating movement into the diagrams would be challenging without the assistance of parametric equations. But, it could be the basis of a problem at a particular point (or set of points) in time.

But, the idea of using epicycles to create complex patterns in the abstract can be introduced at an earlier level through the use of a Spirograph (Andrew, 2009). There is an online version at Inspiral Web (n.d.).

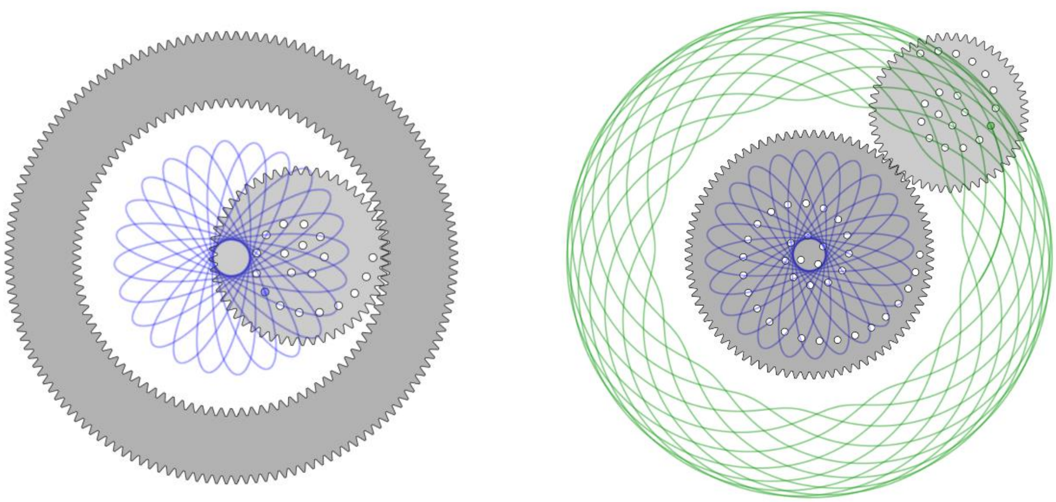


Figure 3. Images from Inspiral Web created by the author.

While a Spirograph doesn't have the same level of flexibility as a parametric function grapher, the Spirograph mimics the effects of epicyclic behavior and a careful selection of components can produce a wide variety of complex paths. At a high level, students might be asked to determine the relative ratios of the equivalent radii, and the frequency and try to use that information to recreate the graphs they make on a parametric equation grapher.

Taken altogether, I think the precalculus students could get the most from these topics. They have enough math to be able to gain real insights, even from something like the Spirograph that they likely played with as a child. Connecting the math to sciences like astronomy and art through the Spirograph, or music, could engage students who might have otherwise tuned out.

I think including the history of math into the classroom does impact student achievement. Students who don't already love math for its own sake, connecting ideas to history and the motivations behind why things are done can help them connect with the material. We all too often teach math in a kind of vacuum that is disconnected from other subjects, so of course, we get the "when am I ever going to use this" question all the time. But if more students could find something that interested them, even if it's just the personalities, they will be more likely to stick with the topic and do what they need to do to succeed.

References

(2021). Retrieved from GraphFree.com: https://www.graphfree.com/grapher.html

Andrew, P. (2009). *The mathematics of the epicycloid : a historical journey with a modern perspective.* University of New Mexico. Retrieved from https://digitalrepository.unm.edu/cgi/viewcontent.cgi?article=1001&context=math_etds

Ball, W. W. (1960). *A Short Account of the History of Mathematics.* New York: Dover Publications, Inc.

Horner, L. (2016). *Epicycles*. Retrieved from Geogebra: https://www.geogebra.org/m/w4JQb9Sj

Inspiral Web. (n.d.). Retrieved from https://nathanfriend.io/inspiral-web/

Ptolemy, C. (1998). *Almagest.* (G. J. Toomer, Trans.) Princeton University Press.

TivnanR. (2018). *Fourier Series Epicycles*. Retrieved from Geogebra: https://www.geogebra.org/m/KgpjCdPT