Ptolemaic Epicycles: Precursor to Fourier Series

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MTH 450

Final Project

Claudius Ptolemy created epicycles to resolve a conflict between two aspects of Greek thought. Following the philosophy of Aristotle, the heavens were thought to be perfect, and therefore could allow only perfectly circular motion (Smith D. , 1953). Second was that the motions of planets in the heavens clearly did not follow paths that could be described by circles. Ptolemy, therefore, proposed a scheme of epicycles (the object follows a circular path around a point, which is itself moving in a circular path, which can be repeated in multiple layers) (Ptolemy, 1998). The scheme, by adjusting the radius and the speed of rotation of the epicycles, one can approximate a wide range of behavior, including motions that included retrograde motion of the planets, elliptical orbits and so forth. Adding layers of epicycles allows the modeling of motions to become as accurate as needed depending on the number of epicycles included. It was this scheme of approximations that dominated the description of planetary motion until Kepler (Sargent, 1917).

Ptolemy did all his work in geometric terms, but if one adopts a view of epicycles using parametric equations, it becomes easy to see that stacking epicycles creates functions for both x and y that depend on sine and cosine functions, exactly the basis for modern Fourier series (Andrew, 2009). In Ptolemy's time, the requirements of circular motion were seen as necessary and real, but taken as an instrumental method of approximations, the scheme remains powerful for making accurate predictions about planetary motion independent of the physical reality of those motions. Separated from requiring a physical explanation, the method of approximations using sine and cosine could then be free to be refined and applied to other kinds of curves independent of heavenly motion. In modern mathematics, Fourier series are clearly billed as an instrumental method of approximation, not as a representation of reality. Fourier series requires functions that it approximates to observe certain behaviors, such as periodicity, precisely the kind of behavior one can expect from orbital motion, and not necessarily from most common function types (Ball, 1960).

To begin to set the stage for Ptolemaic epicycles, we begin by considering the Aristotelian physical world that Claudius Ptolemy hoped to explain, and within whose constraints, his model needed to apply. Aristotle, as described in the *Physics*, believed that the universe was made up of two types of matter: the celestial and the terrestrial. Celestial matter was composed of a fifth element, which he called aether (ether) and which is sometimes called quintessence, and was perfect, immutable, and eternal. The terrestrial matter, on the other hand, was composed of the four classical elements - earth, water, air, and fire - and was subject to change, decay, and corruption. Aristotle further argued that the celestial bodies, such as the stars and planets, were made of this perfect and immutable aether, and *that they moved in perfect circular orbits around the Earth* [emphasis mine]. Earthly matter could only be in the center and thus, required a geocentric worldview. He believed that the perfection of the heavens was demonstrated by their regular and uniform motion, which was a reflection of their perfect and unchanging nature. He also believed that the celestial bodies were eternal and uncreated, and that they existed outside of time and space (Aristotle, 1957).

Aristotle's philosophy about the perfection of the heavens was the dominant view of the celestial realm for many centuries, and so it was within this paradigm that Greek astronomy and cosmology developed. Aristotle's ideas about the perfection of the heavens and the uniform circular motion of the celestial bodies provided a theoretical framework for the development of astronomical models using epicycles (Ptolemy, 1998). An epicycle is a small circle whose center moves along the circumference of a larger circle. Epicycles were used by ancient astronomers to explain the observed retrograde motion of the planets, which appeared to move backwards in the sky at certain times. Aristotle's model of the universe, which placed the Earth at the center and had the planets moving in perfect circular orbits around it, did not account for retrograde motion (suggesting that he may have been unaware of it). However, his ideas about the perfection of the heavens had to be accounted for in any model of planetary motion, as well as conforming to the reality of observations. This system of nested circles and epicycles allowed astronomers to explain retrograde motion while preserving the idea of uniform circular motion, at least in the abstract (Sargent, 1917).

The use of epicycles in astronomical models continued to evolve over time, with different astronomers proposing different numbers and configurations of epicycles, and not all of them conformed strictly to the requirements of uniform circular motion; they stretched Aristotle's physics but tried not to break it completely. The most famous example is the geocentric model of the universe proposed by Ptolemy in the 2nd century CE, which used a complex system of epicycles to account for the observed motion of the planets. Ptolemy used a number of devices in his model of epicycles, including non-uniform circular motion, to account for the behavior of the planets. This model remained the dominant view of the universe for centuries, and was only replaced by the heliocentric model of Copernicus in the 16th century. Nonetheless, the use of epicycles in astronomical models owes a debt to Aristotle's ideas about the perfection of the heavens and the uniform circular motion of the celestial bodies (Cajori, 1897).

Let's examine Ptolemaic epicycles in greater detail. In the *Almagest*, Ptolemy used a system of epicycles to model the motion of the planets. Figure 1 shows the basics of the system and most common terms.

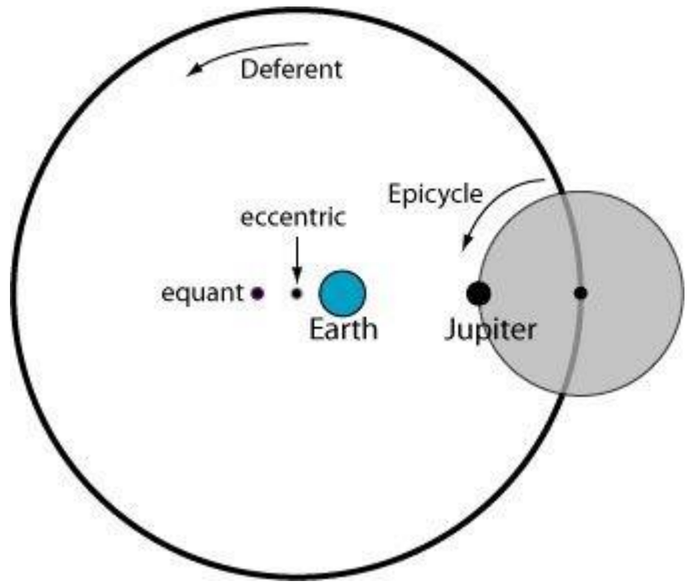

Figure 1. A simplified model of the elements of Ptolemaic epicycles (2020).

In Greek astronomy, the terms deferent, equant, eccentric, and epicycle refer to the components of the Ptolemaic model used to explain the motion of planetary bodies around the Earth. Refer to the image in Figure 1 for their place in the Ptolemaic system.

Deferent: The deferent is a circle that represents the orbit of a planet around the Earth. The center of the deferent is located at a point called the eccentric, which is displaced from the Earth's center. The planet moves uniformly around the deferent.

Equant: The equant is a point inside the deferent circle that is not the Earth's center. The planet moves at a uniform angular speed around the equant. This means that the planet moves faster when it is closer to the Earth, and slower when it is farther away. Motion around the Earth or the center of the deferent is not uniform.

Eccentric: The eccentric is a point that is off-center from the Earth's center within the deferent circle. The planet moves around the eccentric in a circular orbit, while the eccentric moves uniformly around the Earth's center.

Epicycle: An epicycle is a smaller circle that is centered on the eccentric point. The planet moves around the epicycle in a circular orbit, while the epicycle moves around the deferent circle, producing more complex behavior than simple circular motion could produce on its own.

The motion of the planet around the deferent is known as the planet's eccentric motion, and the motion of the deferent around the Earth is known as the planet's epicyclic motion. The combination of these two circular motions creates a complex path for the planet that accounts for its observed motion in the sky, including retrograde motion. All of these elements (and additional layers of epicycles) could be used in combination to better approximate the motion of the planets than could a naïve circular orbit as envisioned by Aristotle. One can see elements of the model that hints at the elliptical motion that was really behind the true motion. Displacing the Earth from the exact center is similar to a feature of elliptical orbits that place the gravitational body at one focus and not the center. Moreover, the constant angular speed allows for additional adjustments to planetary motion that also mimic elliptical orbits that also move along its orbit at varying speeds. These features, some of which violated the strict letter of Aristotelian physics, allowed Ptolemy to adjust the speed and direction of the planet's motion at different points in its orbit (Ptolemy, 1998).

The geometry of epicycles used by Ptolemy in the *Almagest* was a complex and sophisticated system that allowed astronomers to predict the positions of the planets with a high degree of accuracy. However, it also required a large number of epicycles and equants, making the model somewhat cumbersome and difficult to use. Nonetheless, the use of epicycles in the *Almagest* represented a major advance in the development of mathematical astronomy, and had a profound influence on the study of astronomy and the natural sciences. The development of the heliocentric model by Copernicus, which placed the Sun at the center of the solar system, eventually replaced the geocentric models that relied on these components. While the Copernican model reduced the number of features required to explain planetary motion, his system still used circles and still needed several layers of epicycles to approximate the observed behavior of the planets (Smith D. , 1951).

Copernicus's heliocentric model of the solar system represented a major departure from the geocentric models that had been used for centuries. In the heliocentric model, the Sun is at the center of the solar system, with the planets, including Earth, orbiting around it in circular paths. In some respects, Copernicus's model was more consistent with certain aspects of Aristotelian physics because it was more closely based on the idea of a simple circular motion, despite abandoning the idea of Earth at the center. The planets were believed to move in circles around the Sun, with their speeds varying according to their distance from the Sun (Ball, 1960).

To explain the apparent retrograde motion of the planets, Copernicus used epicycles similar to those used in the Ptolemaic system. However, in Copernicus's system, the epicycles were centered on the Sun rather than on the Earth. This simplified the model and eliminated the need for equants and deferents. Copernicus's model was still not entirely accurate, but it provided a much simpler and more elegant explanation of the motion of the planets than the Ptolemaic system. His ideas paved the way for later astronomers, such as Johannes Kepler and Galileo Galilei, to refine the heliocentric model and develop a more accurate understanding of the solar system, and paved the way for an eventual abandonment of Aristotelian physics (Andrew, 2009).

One of the main controversies that eventually drove astronomers to break away from the Ptolemaic system, and Aristotelian physics in general, is the question of scientific realism vs. mere instrumentalism. Realism and instrumentalism are two philosophical positions regarding the nature of scientific theories. Realists argue that scientific theories are true descriptions of the world (or should be as much as possible), while instrumentalists argue that scientific theories are simply tools for making predictions and should not be taken as literally true. The controversy surrounding the epicycle models of the heavens was rooted in this basic debate. Realists argued that the epicycle models accurately described the motion of the planets, and that the deferents, epicycles, and other complex features of the models were necessary to accurately capture the true complexity of the heavens. They believed that the models represented the actual structure of the universe and that they provided a true understanding of the motion of the planets. Instrumentalists, on the other hand, argued that the epicycle models were simply tools for making predictions, and that their accuracy did not necessarily imply that they were true descriptions of the universe. They believed that the models were useful for making predictions, but that their complexity made them difficult to understand and interpret in any literal sense (Chakravartty, 2017).

The controversy was further complicated by the fact that the models were often associated with particular philosophical or religious beliefs. For example, some realists believed that the complexity of the epicycle models reflected the inherent complexity of the universe, which they saw as a reflection of the mind of God. Instrumentalists, meanwhile, saw the models as a way of understanding the universe in a purely scientific way, without reference to any particular philosophical or religious beliefs. It can be argued that the break with the Ptolemaic system was that many scientists saw the epicycle model as instrumentally accurate for predicting the positions of the planets, and yet, at the same time, they desired a model that could be seen as actually representative of what was happening in reality, which the epicycle model did not satisfy. This conflict drove the desire for a model that was at least as good as predicting positions of the planets, but was more likely to be an accurate model of reality (Chakravartty, 2017).

In order to more closely examine the mathematics of the theory of epicycles, we're going to look at a simplified version of the theory that omits some of the more complex elements such as equants and non-uniform motion. The theory of epicycles can be represented as parametric equations by defining two sets of coordinates: one for the center of the epicycle and one for the planet itself. Parametric equations are anachronistic for Ptolemy, but are basic elements of modern mathematics that allow us to consider their behavior.

Let us assume that the center of the epicycle is located at the point (a, b) in a two-dimensional coordinate system, and that the planet is located at the point (x, y) on the epicycle. Let us also assume that the epicycle has radius r_1, and that the planet travels around the epicycle with angular velocity ω_1. Then, the position of the planet can be described by the following set of parametric equations:

$$\begin{cases} x(t) = a + r_1 cos(\omega_1 t) \\ y(t) = b + r_1 sin(\omega_1 t) \end{cases} \quad (1)$$

where t is the time parameter, which can be used to calculate the position of the planet at any given time. These equations describe the motion of the planet in terms of its distance from the center of the epicycle and its angle relative to a fixed reference point.

The parameters a, b, r_1, and ω_1 can be adjusted to model different epicycle systems, allowing for a wide range of possible planetary motions. By using parametric equations to describe the motion of the planets, astronomers and mathematicians were able to accurately predict the positions of the planets in the sky, despite the apparent complexity of their motions. Moreover, if the position of the deferent at the point (a, b) are both themselves functions of time and move in a circle, then they, too, are functions of sine and cosine, with its own angular velocity ω_0, and its own radius r_0. The system then becomes

$$\begin{cases} x(t) = r_0 \cos(\omega_0 t) + r_1 \, cos(\omega_1 t) \\ y(t) = r_0 \sin(\omega_0 t) + r_1 \, sin(\omega_1 t) \end{cases} (2)$$

Alternatively, depending on the direction of motion, we might also have:

$$\begin{cases} x(t) = r_0 \sin(\omega_0 t) + r_1 \, cos(\omega_1 t) \\ y(t) = r_0 \cos(\omega_0 t) + r_1 \, sin(\omega_1 t) \end{cases} (3)$$

In either version of the model, the position at any time t is modeled by combinations of sine and cosine functions. An example is shown in Figure 2.

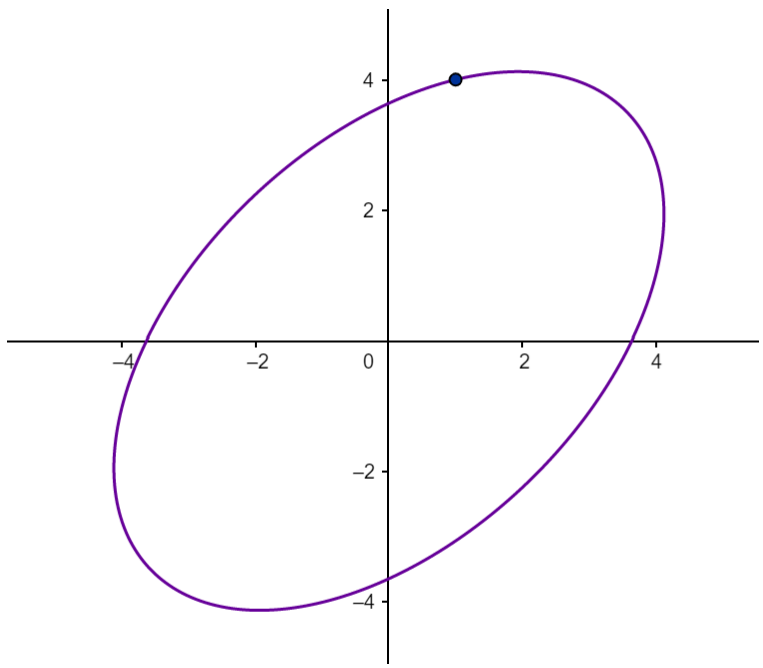

Figure 2. Model using two epicycles of $x(t) = 4\sin(t) + \cos(t)$, $y(t) = 4\cos(t) + \sin(t)$ Which produces an ellipse. Image produced by the author. (McCall, 2023)

By adjusting the radii of the two circles and their angular velocity, complex behavior like retrograde motion is possible. Figure 3 shows an example of such retrograde behavior as seen from the perspective of someone in the center of the figure.

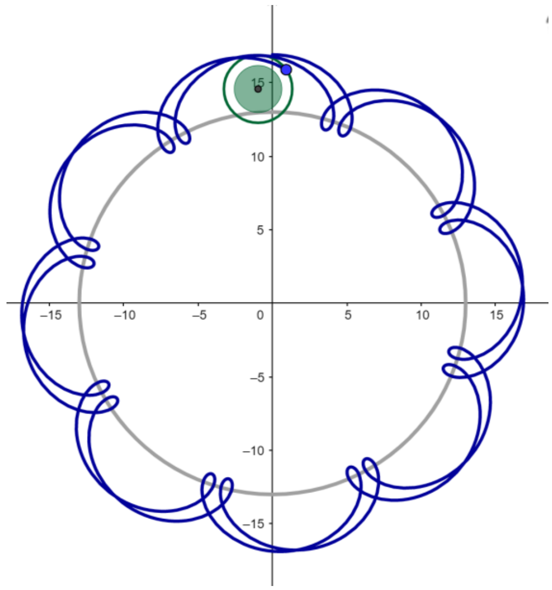

Figure 3. The system shown is $x(t) = 14.6\sin(0.07t) + 2.3\sin\left(\frac{1}{1.6}t\right)$, $y(t) = 14.6\cos(0.07t) + 2.3\cos\left(\frac{1}{1.6}t\right)$, with t plotted on the interval [0,182.5] (Horner, 2016).

We can see the relationship between epicycles and Fourier series more clearly when we stretch out the model in one dimension and plot it against time. An animation snapshotted in Figures 4 and 5 illustrates the connection with multiple epicycles.

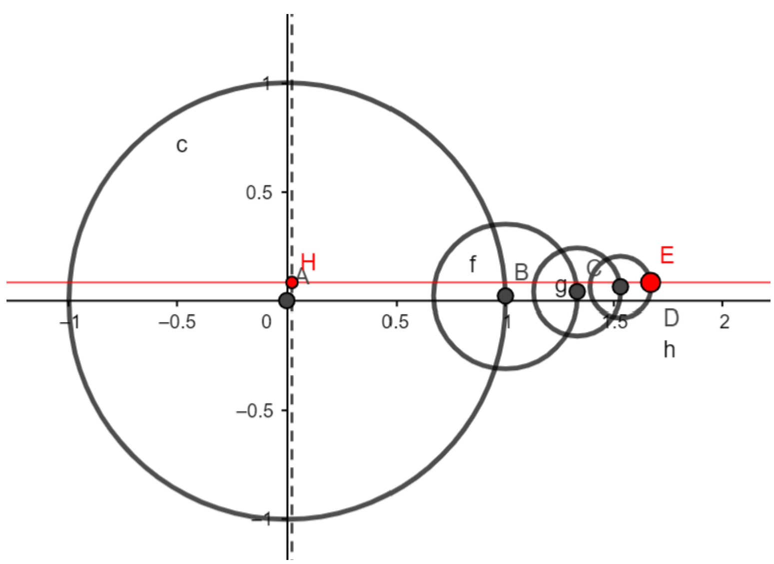

Figure 4. The 3-epicycle system at the starting point (TivnanR, 2018).

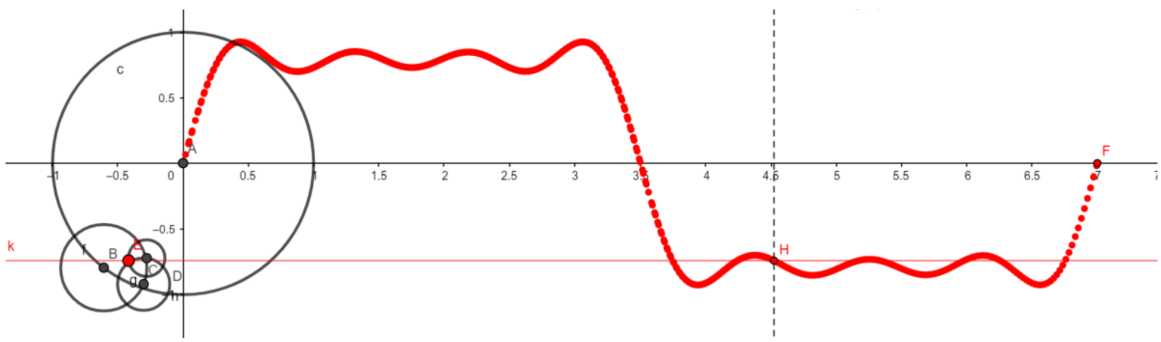

Figure 5. A complete cycle of the 3-epicycle model using default parameters on the website. These parameters can be adjusted to obtain different behavior patterns (TivnanR, 2018).

Fourier series represent functions as a sum of trigonometric functions. They are named after the French mathematician Joseph Fourier, who developed the theory in the early 19th century. However, the concepts behind Fourier series can be traced back to the 18th century work of mathematicians Brook Taylor, Jean le Rond d'Alembert, and Euler. Taylor developed a way of representing any function as an infinite sum of polynomial functions, known as the Taylor series. This work was further developed by d'Alembert (Ball, 1960).

The specific idea of representing functions using Fourier series was first introduced by Fourier in his work on heat transfer. He showed that any periodic function can be represented as a sum of sines and cosines, with the coefficients determined by the function's Fourier series.

There is a connection between the theory of epicycles and Fourier series approximations, as both involve representing a periodic function as a combination of simpler functions. In the case of epicycles, the motion of a planet is modeled as a combination of circular motions with different frequencies and amplitudes. This can be seen as a way of approximating the planet's motion as a sum of simpler, periodic motions. Similarly, in Fourier series approximations, a periodic function is represented as a sum of simpler trigonometric functions (sine and cosine) with different frequencies and amplitudes. This allows the function to be approximated by a finite number of terms, which can be used to compute its values at different points in time. When we examine epicycles in terms of parametric functions, we see the clear connection between the inherently periodic motion of the planets and employing combinations of sine and cosine functions to approximate complex behavior. Once Ptolemaic epicycles were clearly freed from realism claims, approximation methods were free to be inspired by these methods.

Consider the Fourier approximation of a function $f(t)$:

$$f(t) = \sum_{n=-\infty}^{n=\infty} c_n (\cos(\frac{n\pi}{L}t) + i\sin(\frac{n\pi}{L}t)) \tag{4}$$

If we take terms from $n = -k$ to $n = k$, we get the following expansion:

$$f(t) \approx c_{-k}(\cos(\frac{k\pi}{L}t) - i\sin(\frac{k\pi}{L}t)) + \ldots + c_{-2}(\cos(\frac{2\pi}{L}t) - i\sin(\frac{2\pi}{L}t)) + c_{-1}(\cos(\frac{\pi}{L}t) - i\sin(\frac{\pi}{L}t)) + c_0$$

$$+ c_1(\cos(\frac{\pi}{L}t) + i\sin(\frac{\pi}{L}t)) + c_2(\cos(\frac{2\pi}{L}t) + i\sin(\frac{2\pi}{L}t)) + \ldots + c_k(\cos(\frac{k\pi}{L}t) + i\sin(\frac{k\pi}{L}t)) \tag{5}$$

As Andrew notes in her dissertation, this is equivalent to $2k - 1$ epicycles with a center at c_0, and c_n are the radii of the corresponding epicycle (2009). One interesting difference to note here is that in Ptolemaic epicycles, the angular frequencies were chosen carefully to obtain particular behaviors and were not necessarily tied to integer multiples of the periods. However, the downside with this method is that the procedure is rooted in trial-and-error to find the best fit. One advantage is that because the parameters are chosen carefully, one can reduce the number of epicycles in the model. The advantage of Fourier series is that they can be chosen systematically, so trial-and-error is not required, but this does leave the model with potentially many more terms.

Moreover, what Fourier series makes clear is that epicycles can be used to model the behavior of any parametric function, even complex ones such as flowers or triangles, if we are willing to continue stacking epicycles (Andrew, 2009).

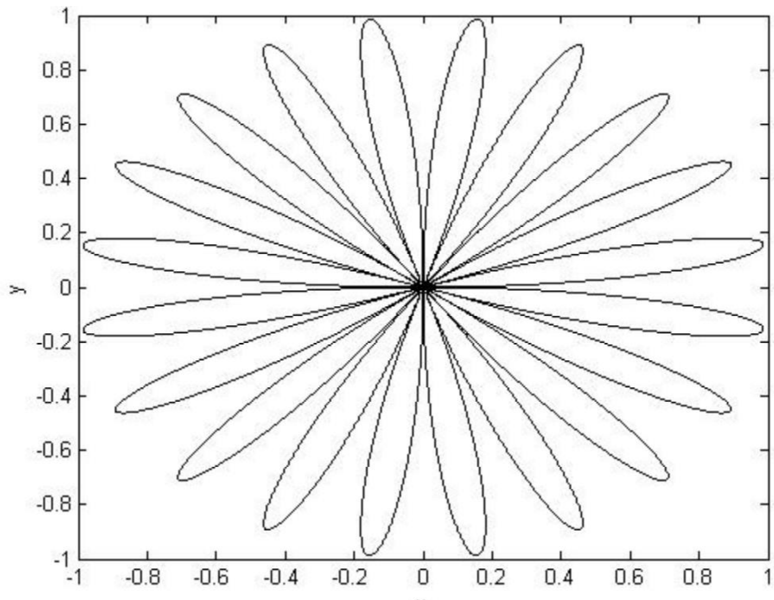

Figure 6. Plot of a Fourier series of a flower (Andrew, 2009)

We can use this relationship between the two types of models to inform our approach to mathematics education. While teaching epicycles would seem to be anachronistic in a modern curriculum where real world applications are often a significant focus. Nonetheless, because of the connection to Fourier series, which are a significant feature of advanced modeling methods used in engineering, introducing them conceptually in a mathematics course can have significant advantages. Some of the ideas behind Fourier series are often introduced with processing sound because sound is naturally a collection of wave functions. Because epicycles are based on relatively simple geometric concepts (circles) and parametric equations, mathematics students in geometry or precalculus courses have the tools to develop a more intuitive understanding of how the model works and make the connection to the Fourier transform techniques even if they don't end up mastering the math behind it.

In an earlier time in US history, teaching the theory of epicycles in a math class would not have been considered appropriate since it was an abandoned scientific theory. However, as we can see from the preceding discussion, the mathematics of it can be quite relevant even if the specific claims of realism for the model are no longer adopted. As Andrew notes in her dissertation, we can even use art as an entry point to the mathematics without the mechanics of it through a Spirograph (2009). Students could even be inspire to write computer models that can produce a variety of motions and paths that could include both planetary motion and artistic design. Students are far more likely to remember the concepts behind the mathematics they learn if they can make a connection to other areas that might interest them more. Connecting trigonometry and parametric equations to music, art and astronomy could go a long way to encouraging students to continue to pursue math in the future.

We see in many areas how the Greeks inspired developments in modern mathematics. Just as Archimedes developed techniques that were later refined with the development of calculus, Ptolemaic epicycles presaged the development of Fourier series. A lot would have to happen to make this possible including the development of trigonometry (sine and cosine functions) and the development of parametric equations. Fourier series further required the development of calculus and differential equations (Jacob & Evans, 2018). Unlike Ptolemaic epicycles, Fourier series never made any claims to realism and so has been far less controversial in history than epicycles, but as approximation methods, both are powerful techniques that remain useful for modern mathematics.

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